1 Basic Definitions.

An $N \times N$ symmetric matrix $A$ is positive definite iff for any $v \neq 0$, $v'Av > 0$. For example, if

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

then the statement is that for any $v = (v_1, v_2) \neq 0$,

$$v'Av = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} av_1 + bv_2 \\ bv_1 + cv_2 \end{bmatrix} = av_1^2 + 2bv_1v_2 + cv_2^2 > 0$$

(The expression $av_1^2 + 2bv_1v_2 + cv_2^2$ is called a quadratic form.) An $N \times N$ symmetric matrix $A$ is negative definite iff $-A$ is positive definite. The definition of a positive semi-definite matrix relaxes $>$ to $\geq$, and similarly for negative semi-definiteness.

If $N = 1$ then $A$ is just a number and a number is positive definite iff it is positive. For $N > 1$ the condition of being positive definite is somewhat subtle. For example, the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

is not positive definite. If $v = (1, -1)$ then $v'Av = -4$. Loosely, a matrix is positive definite iff (a) it has a diagonal that is positive and (b) off diagonal terms are not too large in absolute value relative to the terms on the diagonal. I won’t formalize this assertion but it should be plausible given the example. The canonical positive definite matrix is the identity matrix, where all the off diagonal terms are zero.

A useful fact is the following. If $S$ is any $M \times N$ matrix then $A = S'S$ is positive semi-definite. To see this, note that $S'S$ is symmetric $N \times N$. To see that it is positive semi-definite, note that for any $v \in \mathbb{R}^N$,

$$v'Av = v'[S'S]v = (v'S')(Sv) = (Sv)'(Sv) \geq 0.$$
Moreover, by this argument, \( v'Av = 0 \) iff \( Sv = 0 \). This cannot happen if \( S \) has full column rank. Thus \( S' \) is positive definite, not merely positive semi-definite, if the rank of \( S \) is \( N \), which implies \( N \leq M \).

The converse of all this is also true, although I will not establish it. If \( A \) is positive definite then there is a full rank \( N \times N \) matrix \( S \) such that \( A = S' \) \( S \). If \( A \) is negative semi-definite and has rank \( M \leq N \) then there is an \( M \times N \) matrix of rank \( M \) such that \( A = S' S \).

### 2 Inverses of Definite Matrices.

**Theorem 1.** If \( A \) is positive definite then \( A \) is invertible and \( A^{-1} \) is positive definite.

**Proof.** If \( A \) is positive definite then \( v'Av > 0 \) for all \( v \neq 0 \), hence \( Av \neq 0 \) for all \( v \neq 0 \), hence \( A \) has full rank, hence \( A \) is invertible.

For any invertible matrix \( A \), \( (A^{-1})' = (A')^{-1} \). To see this, note that \( [A'(A^{-1})]' = A^{-1}A = I \). Hence \( A'(A^{-1})' = I' = I \). Similarly, \( (A^{-1})'A' = I \).

If \( A \) is symmetric and invertible then \( (A^{-1})' = (A')^{-1} = A^{-1} \), hence \( A^{-1} \) is symmetric. It remains to show that \( A^{-1} \) is positive definite. Consider \( v'A^{-1}v \) for any \( v \neq 0 \). Then \( v'A^{-1}v = v'A^{-1}AA^{-1}v = (A^{-1}v)'A(A^{-1}v) > 0 \), where the second equality comes from the fact that \( A^{-1} \) is symmetric and the last inequality follows from the fact that \( A \) is positive definite. ■

Likewise, if \( A \) is negative definite then \( A^{-1} \) exists and is negative definite.

### 3 Checking Definiteness.

There is a mechanical check for definiteness of a matrix. It is messy but easy to implement on a computer.

Given an \( N \times N \) matrix \( A \), a **leading principal submatrix** of \( A \) is a submatrix formed by deleting all but the first \( n \) rows and columns. Thus, if

\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix},
\]

then the leading principal submatrices are

\[
\begin{bmatrix}
a_{11}
\end{bmatrix},
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix},
\]

and \( A \) itself. A **leading principal minor** is the determinant of a leading principal submatrix.
Theorem 2. A matrix is positive definite iff all of the leading principal minors are positive.

Proof. Omitted. ■

Since $A$ is negative definite iff $-A$ is positive definite, and since multiplying an $n \times n$ matrix by -1 multiplies the determinant by $(-1)^n$, Theorem 2 implies that a matrix is negative definite iff the principal minors alternate in sign, with the sign negative iff the number of columns in the submatrix of $A$ is odd.

Note that Theorem 2 implicitly assumes that the matrix is symmetric. If the matrix is not symmetric then the leading principal minor condition does not guarantee that $v'Av > 0$ for all $v \neq 0$. For example, consider

$$A = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix},$$

This matrix passes the principal minor test, but for $v = (1, -1)$, $v'Av = -8$. 