John Nachbar
Washington University in St. Louis
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Fixed Point Theorems[1]

1 Overview

Definition 1. *Given a set X and a function $f : X \to X$, $x^* \in X$ is a* fixed point *of f iff $f(x^*) = x^*$.*

Many existence problems in economics – for example existence of competitive equilibrium in general equilibrium theory, existence of Nash in equilibrium in game theory – can be formulated as fixed point problems. Because of this, theorems giving sufficient conditions for existence of fixed points have played an important role in economics.

My treatment is schematic, focusing on only a few representative theorems and omitting some proofs. An excellent introduction to fixed point theory is Border (1985). McLennan (2008) is a more recent concise survey that provides a treatment more sophisticated than the one here. McLennan also has a book-length treatment; a draft is here.

The remainder of these notes is divided into three sections. Section 2 focuses on fixed point theorems where the goal is to find restrictions on the set X strong enough to guarantee that *every* continuous function on X has a fixed point. The prototype of theorems in this class is the Brouwer Fixed Point Theorem, which states that a fixed point exists provided X is a compact and convex subset of $\mathbb{R}^N$. In contrast, the Contraction Mapping Theorem (Section 3) imposes a strong continuity condition on f but only very weak conditions on X. Finally, the Tarski Fixed Point Theorem (Section 4) requires that f be weakly increasing, but not necessarily continuous, and that X be, loosely, a generalized rectangle (possibly with holes).

2 The Brouwer Fixed Point Theorem and its Relatives

2.1 The Brouwer Fixed Point Theorem in $\mathbb{R}$.

Theorem 1. *If $X = [a, b] \subseteq \mathbb{R}$ and $f : X \to X$ is continuous then f has a fixed point.*

Proof. If $f(a) = a$ or $f(b) = b$ then we are done. Otherwise, $f(a) > a$ and $f(b) < b$. Define $g(x) = f(x) - x$. Then $g(a) > 0$ while $g(b) < 0$. Moreover, g is continuous

 Michael Greinecker pointed out a mistake in the statement of Eilenberg-Montgomery in a previous version. That error is now corrected.

since f is continuous. Therefore, by the Intermediate Value Theorem, there is an $x^* \in (a,b)$ such that $g(x^*) = 0$, hence $f(x^*) = x^*$. ■

The following are examples in which one of the sufficient conditions in Theorem 1 are violated and no fixed point exists.

Example 1. Let $X = [0,1)$ and $f(x) = (x+1)/2$. There is no fixed point. Here, f is continuous and X is connected, but X is not compact. □

Example 2. Let $X = [0,1]$ and

$$f(x) = \begin{cases} 1 & \text{if } x < 1/2, \\ 0 & \text{if } x \geq 1/2. \end{cases}$$

There is no fixed point. Here X is connected and compact but f is not continuous. □

Example 3. Let $X = [0,1/3] \cup [2/3,1]$ and $f(x) = 1/2$. There is no fixed point. Here f is continuous and X is compact, but X is not connected. □

Finally, note that the conditions in Theorem 1 are sufficient but not necessary. The requirement that a fixed point must exist for *every* continuous f imposes much stronger conditions on X than the requirement that a fixed point exist for some given f. For given X and f, a fixed point may exist as long as X is not empty, even if every other condition is violated.

Example 4. Let $X = (0,1/3) \cup (2/3,\infty)$. Let

$$f(x) = \begin{cases} 1/4 & \text{if } x = 1/4 \\ 0 & \text{otherwise.} \end{cases}$$

Then X is not closed, not bounded, an not connected and f is not continuous. But f has a fixed point, namely $x^* = 1/4$. □

The question now is how to generalize Theorem 1 from a statement about $X \subseteq \mathbb{R}$ to a statement about $X \subseteq \mathbb{R}^N$. There are two issues.

The first issue is that the line of proof in Theorem 1 does not generalize to higher dimensions. For $X = [0,1]$, a fixed point occurs where the graph of f crosses the 45° line. Since the 45° line bisects the square $[0,1]^2 \subseteq \mathbb{R}^2$, if f is continuous then its graph must cross this line; the proof based on the Intermediate Value theorem formalizes exactly this intuition. In contrast, if $X = [0,1]^2 \subseteq \mathbb{R}^2$, then the graph of f lies in the 4-dimensional cube $[0,1]^2 \times [0,1]^2$, and the analog of the 45° line is a 2-dimensional plane in this cube. A 2-dimensional plane cannot bisect a 4-dimensional cube (just as a 1-dimensional line cannot bisect a 3-dimensional cube). Brouwer must, among other things, insure that the graph of f, which is 2-dimensional, does not spiral around the 2-dimensional 45° plane, without ever intersecting it.

The second issue is that it is not obvious how to generalize the condition that X be an interval. Requiring X to be a closed rectangle is too strong. On the other hand, requiring X to be compact and connected is too weak, as the next example illustrates.

Example 5. Let X be a disk with a central hole cut out

$$X = \{x \in \mathbb{R}^2 : \|x\| \in [\varepsilon, 1]\}$$

where $\varepsilon \in (0, 1)$. Then X is compact and connected but it is not convex. Let f be the function that, in effect, rotates X by a half turn. More formally, represent $\mathbb{R}^2$ in polar coordinates: a point (r, θ) corresponds to $x_1 = r\cos(\theta)$, $x_2 = r\sin(\theta)$. Then $f(r, \theta) = (r, \theta + \pi)$. This function is continuous but there is no fixed point. □

In this example, X is connected but not convex, which leads naturally to the conjecture that a fixed point exists if X is compact and convex. This intuition is correct, but convexity can be weakened, at essentially no cost, for a reason discussed in the next section.

2.2 Homeomorphisms and the Fixed Point Property.

Definition 2. *A non-empty metric space* (X, d) *has the* fixed point property *iff for any continuous function* $f : X \to X$, f *has a fixed point.*

Definition 3. *Let* (X, d_x) *and* (Y, d_y) *be metric spaces.* X *and* Y *are* homeomorphic *iff there exists a bijection* $h : X \to Y$ *such that both* h *and* h^{-1} *are continuous.*

Say that a property of sets/spaces is *topological* iff for any two homeomorphic spaces, if one space has the property then so does the other. Elsewhere, I have shown that compactness and connectedness are both topological properties. In $\mathbb{R}$, convexity is topological because, in $\mathbb{R}$, convexity is equivalent to connectedness. But, more generally, convexity is not topological. For example, in $\mathbb{R}^2$, a figure shaped like a five-pointed star is not convex even though it is homeomorphic to a convex set, a pentagon for example. On the other hand, the fixed point property is topological.

Theorem 2. *Let* (X, d_x) *and* (Y, d_y) *be non-empty metric spaces. If* X *and* Y *are homeomorphic and* X *has the fixed point property then* Y *also has the fixed point property.*

Proof. Suppose that X has the fixed point property, that $h : X \to Y$ is a homeomorphism, and that $g : Y \to Y$ is continuous. I need to show that g has a fixed point. Define $f = h_{-1} \circ g \circ h$. Then $f : X \to X$ is continuous, as a composition of continuous functions. Since X has the fixed point property, there is a point $x^* \in X$ such that $f(x^*) = x^*$, meaning $h^{-1}(g(h(x^*)) = x^*$, or $g(h(x^*)) = h(x^*)$, which implies that $h(x^*)$ is a fixed point of g. ■

2.3 The Brouwer Fixed Point Theorem

Let

$$\Delta^N = \left\{ x \in \mathbb{R}_+^{N+1} : \sum_n x_n = 1 \right\}.$$

Δ^N is a special case of a regular N-simplex. More generally, a regular N-simplex is defined by $N+1$ equally spaced points, the *vertices* of the simplex. In the case of Δ^N the vertices are $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, $\ldots$, $(0, \ldots, 0, 1)$, all of which are distance 1 apart. A regular 1-simplex is a line segment. A regular 2-simplex is an equilateral triangle. A regular 3-simplex is a regular tetrahedron. Any regular N-simplex is (almost trivially) homeomorphic to Δ^N.

Theorem 3. *Δ^N has the fixed point property.*

Proof. Fix a continuous function $f : \Delta^N \to \Delta^N$.

For each $t \in \mathbb{N}$, $t \geq 1$, there is a *simplicial subdivision* of Δ^N, formed by introducing vertices at the points

$$\left(\frac{k_1}{t}, \ldots, \frac{k_{N+1}}{t} \right),$$

where $k_n \in \mathbb{N}$ and $\sum_n k_n/t = 1$. Note that this set of vertices is finite and includes the original vertices. The closest of these vertices are distance $1/t$ apart and any set of $N+1$ such closest vertices defines an N-simplex that is a miniature version of Δ^N. Call these new simplexes *sub-simplexes.* For example, if $N = 2$ and $t = 2$, then three new vertices are introduced, at $\{1/2, 1/2, 0\}$, $\{1/2, 0, 1/2\}$, and $\{0, 1/2, 1/2\}$, for a total of six vertices. These six vertices divide the original simplex into four sub-simplexes, each with sides of length 1/2. If $t = 3$, the simplex is divided into nine sub-simplexes, each with sides of length 1/3.

If for any subdivision of Δ^N there is a fixed point at some vertex $v \in \Delta^N$ then we are done. Suppose then that for every t, no vertex is a fixed point.

By a *labeling* of a simplex I mean a function that assigns a number in $n \in \{1, \ldots, N+1\}$ to each vertex. Say that a sub-simplex is *completely labeled* iff each of its $N+1$ vertices has a different label.

The proof now proceeds in two steps.

- **Step One.** For each t consider any labeling such that if the label of vertex v is n then $f_n(v) < v_n$. Since $f(v) \neq v$ (by assumption, no vertex is a fixed point) and since $\sum_n f_n(v) = 1 = \sum_n v_n$, there must be at least one n for which $f_n(v) < v_n$, hence the labeling is well defined. (There may be more than one n for which $f_n(v) < v_n$; in such cases, any such n can be the label for v.)

  Suppose that for each t, Δ^N has a at least one completely labeled sub-simplex. Step Two shows that this is true. Let the $N+1$ vertices of this sub-simplex be $v^1, \ldots, v_t^{N+1}$, where v_t^n has label n.

The point $(v_t^1, \ldots, v_t^{N+1})$ lies in $\Delta^N \times \cdots \times \Delta^N$ ($N+1$ times), which is compact since Δ^N is compact. Therefore, there is a point $(x^{1*}, \ldots, x^{(N+1)*})$ and a subsequence along which $(v_t^1, \ldots, v_t^{N+1})$ converges to $(x^{1*}, \ldots, x^{(N+1)*})$. For each t, the vertices of any sub-simplex are exactly $1/t$ apart. Therefore, for any $\varepsilon > 0$, for all t large enough, for any any n, v_t^n is within ε of x^{1*}. This implies $x^{1*} = \cdots = x^{(N+1)*}$. Call this common limit x^*.

For each t and n, since the label on v_t^n is n,

$$f_n(v_t^n) < v_{tn}^n.$$

Therefore, taking the limit, since f is continuous,

$$f_n(x^*) \leq x_n^*.$$

If any inequality is strict, then $\sum_n f_n(x^*) < \sum_n x_n^*$. But $f : \Delta^N \to \Delta^N$, hence $\sum_n f_n(x^*) = \sum_n x_n^* = 1$. Hence $f_n(x^*) = x_n^*$ for all n: x^* is a fixed point of f, as was to be shown.

- **Step Two.** Fix any Δ^N and any t simplicial subdivision. Consider any labeling of the vertices such that if the label of vertex v is n then $v_n > 0$. Note that this property was satisfied by the labeling in Step One. Otherwise, the labeling is unrestricted.

Theorem 4 (Sperner's Lemma)**.** *Given Δ^N, a t simplicial subdivision, and a labeling as above, the number of completely labeled sub-simplexes is odd.*

In particular, Sperner's Lemma implies that there is at least one completely labeled subsimplex, which is what is needed to complete the proof of the Brouwer Fixed Point Theorem. The proof of Sperner's Lemma is by induction on N.

If $N = 0$ then $\Delta^N = 1$. The "simplex" is just the point $x = 1$; for any t, the simplicial subdivision is vacuous and the point is completely labeled (the label is 1).

Consider now Δ^N, $N \geq 1$. This simplex has $N+1$ faces defined by any set of N of the vertices. Each face is itself a copy of Δ^{N-1}. Thus, for example, Δ^3, which is a regular tetrahedron, has four faces, each of which is a copy of Δ^2, which is an equilateral triangle.

Any t simplicial subdivision of the original Δ^N induces a t simplicial subdivision on each face. By the induction hypothesis, any face of Δ^{N+1}, being a copy of Δ^N, has an odd number of completely labeled (sub-) subsimplexes.

Given Δ^N, consider the N-dimensional plane containing Δ^N, namely $\{x \in \mathbb{R}^{N+1} : \sum_n x_n = 1\}$. For each t, one can construct a simplicial division of the plane that includes the t simplicial subdivision of the original Δ^N. This simplicial subdivision has vertices at

$$\left(\frac{k_1}{t}, \ldots, \frac{k_N}{t}\right),$$

for $k_n \in \mathbb{Z}$, $\sum_n k_n/t = 1$. For any face of the original Δ^N, consider any subsimplex of this face. This sub-subsimplex is a face shared by two subsimplexes, one a subsimplex of the original simplex and one that is not. Call this latter subsimplex an *exterior subsimplex*.

For any t, label Δ^N as above and consider the following sets.

- S_1. The set of completely labeled subsimplexes of the original Δ^N.
- S_2. The set of subsimplexes of the original Δ^N that have labels $\{1, \ldots, N\}$ but that are missing label $N+1$ (and hence have one of the other labels repeated).
- S_3. The set of exterior subsimplexes for which the face that is a (sub-)subsimplex of the original Δ^N is completely labeled with labels $\{1, \ldots, N\}$ (not label $N+1$).

Let $S = S_1 \cup S_2 \cup S_2$. Let E be the set of subsimplicial faces of Δ^N that are completely labeled with labels $\{1, \ldots, N\}$ (not label $N+1$). Note that for any $e \in E$, e is a face shared by two subsimplexes in S. And any $s \in S$ has at least one face in E.

For $s \in S$, let $\deg(s)$ equal the number of faces in E. One can verify that, independently of N,

- For $s \in S_1 \cup S_3$, $\deg(s) = 1$.
- For $s \in S_2$, $\deg(s) = 2$.

Then

$$\sum_{s \in S} \deg(s) = 2\#E,$$

since each $e \in E$ is a face of two adjoining elements of S, and any $s \in S_1 \cup S_3$ (which has degree 1) has only one face in E, while any $s \in S_2$ (which has degree 2) has two faces in E. This establishes that $\sum_{s \in S} \deg(s)$ is even. On the other hand,

$$\sum_{s \in S} \deg(s) = \#S_1 + 2\#S_2 + \#S_3.$$

By the induction hypothesis, the number of completely labeled (sub-)subsimplexes on any face of Δ^N is odd, hence $\#S_3$ is odd. Since $\sum_{s \in S} \deg(s)$ is even, $2\#S_2$ is even, and $\#S_3$ is odd, it follows that $\#S_1$ is odd, as was to be shown.

■

See Scarf (1982) for how a related proof can be turned into an algorithm for finding fixed points.

An interesting feature of the above proof is that it establishes the existence of an odd number of approximate fixed points (the completely labeled subsimplexes). A natural conjecture is that the number of fixed points must, therefore, be odd, but this is not true.

Example 6. For $X = [0, 1]$, $f(x) = x$ has an infinite number of fixed points. □

Example 7. For $X = [0, 1]$, $f(x) = -6x^3 + 9x^2 - 3x + 4/9 = -(-1 + 6x)(-2 + 3x)^2/9 + x$, which has two fixed points, one at $x = 1/6$ and the other at $x = 2/3$. □

These examples turn out to be pathological: in a sense that can be formalized, if X is homeomorphic to a compact, convex set then "nearly every" continuous function on X has an odd number of fixed points.

A corollary of Theorem 3 is that any compact, convex subset of a Euclidean space has the fixed point property. This is the form in which Brouwer is typically stated.

Theorem 5 (Brouwer Fixed Point Theorem)**.** *If $A \subseteq \mathbb{R}^N$ is non-empty, compact and convex then it has the fixed point property.*

Proof. Since A is bounded, it is a subset of a sufficiently large regular N-simplex, call it S, which in turn is homeomorphic to Δ^N. Consider any continuous $f : A \to A$.

For each $x \in S$, consider the problem $\min_{a \in A} d(x, a)$. This problem has a solution, since A is compact and Euclidean distance is continuous. I claim that the solution is unique. Consider any solutions $a, b \in A$. Let γ note the minimum distance from A to x; then $d(x, a) = d(x, b) = \gamma$.

I will show that $a = b$. Let $c = \frac{1}{2}a + \frac{1}{2}b$. Then $c \in A$, since A is convex, and so,

$$\begin{aligned}
\gamma &\leq d(x, c) \\
&= \left\| x - \left(\frac{1}{2}a + \frac{1}{2}b \right) \right\| \\
&= \left\| \left(\frac{1}{2}x - \frac{1}{2}a \right) + \left(\frac{1}{2}x - \frac{1}{2}b \right) \right\| \\
&\leq \left\| \frac{1}{2}x - \frac{1}{2}a \right\| + \left\| \frac{1}{2}x - \frac{1}{2}b \right\| \\
&= \frac{1}{2}d(x, a) + \frac{1}{2}d(x, b) \\
&= \gamma.
\end{aligned}$$

Hence $d(x, c) = \gamma$. Thus, the second inequality above is an equality, which implies that $x - a$ and $x - b$ are positively collinear (see Remark 1 in the Vector Spaces I notes), which implies (since $\|x-a\| = \|x-b\|$) that $x - a = x - b$, in which case $a = b$, as was to be shown.

For each $x \in S$, let $\phi(x)$ be the unique solution to $\min_{a \in A} d(x, a)$. I claim that $\phi : S \to A$ is continuous. This is a special case of the Theorem of the Maximum. To keep these notes somewhat self-contained, here is a proof. Consider any $x^* \in S$ and any sequence (x_t) in S such that $x_t \to x^*$. I need to show that $\phi(x_t) \to \phi(x^*)$. To simplifiy notation, let $a_t = \phi(x_t)$. Take any convergent subsequence (a_{t_k}), converging to, say, a^*. (A convergent subsequence must exist since A is compact, but I actually don't need this fact at this point in the proof.) I claim that $a^* = \phi(x^*)$ and that $a_t \to a^*$. Consider first any $x \in S$ such that $d(x^*, x) < d(x^*, a^*)$. Then by continuity of d, for t_k large enough, $d(x_{t_k}, x) < d(x_{t_k}, a_{t_k})$. Since $a_{t_k} \in \phi(x_{t_k})$, this implies $x \notin A$. By contraposition, if $a \in A$, then $d(x^*, a) \geq d(x^*, a^*)$, hence $a^* = \phi(x^*)$. Therefore, every convergent subsequence of (a_t) converges to the same point, namely $a^* = \phi(x^*)$; since A is compact, this implies (see the notes on Compactness) that $a_t \to a^*$, hence $\phi(x_t) \to \phi(x^*)$.

Define $g : S \to A$ by $g = f \circ \phi$. Note that for $a \in A$, $\phi(a) = a$, hence $g(a) = f(\phi(a)) = f(a)$. g is continuous since ϕ and f are continuous. Since S is homeomorphic to Δ^N, S has the fixed point property. Therefore, g has a fixed point, x^*, in S: $g(x^*) = x^*$. Since $g(x) \in A$ for every $x \in S$, $x^* \in A$. Since $g(a) = f(a)$ for every $a \in A$, x^* is a fixed point of f. ■

Finally, Theorem 2 and Theorem 5 together imply the following.

Theorem 6. *If $X \subseteq \mathbb{R}^N$ is non-empty and homeomorphic to a compact, convex set then it has the fixed point property.*

2.4 The Kakutani Fixed Point Theorem

A *correspondence* on a set X is a function from X to the set of subsets of X. In notation, $f : X \to \mathbb{P}(X)$. There is a convention to allow $\emptyset$ as a possible value for correspondences, in which case one must specify that the correspondence is non-empty valued in order to rule this out.

Say that $x^* \in X$ is a fixed point of $f : X \to \mathbb{P}(X)$ iff $x^* \in f(x^*)$.

Example 8. Let $X = [0, 1]$ and let

$$f(x) = \begin{cases} 1 & \text{if } x < 1/2, \\ [0,1] & \text{if } x = 1/2, \\ 0 & \text{if } x > 1/2. \end{cases}$$

Then $x^* = 1/2$ is a fixed point of f. □

The Kakutani Fixed Point Theorem, Kakutani (1941), which Kakutani developed with economic applications in mind, extends the Brouwer Fixed Point Theorem to handle correspondences. For example, the proof of existence of Nash equilibrium in finite games can be done just using Brouwer, but that proof requires some effort and uses an auxiliary construction. In contrast, the proof based on the Kakutani Fixed Point Theorem is almost immediate.

To avoid additional complexity, I state the Kakutani Fixed Point Theorem in its original convex form. As with the Brouwer Fixed Point Theorem, convexity can be relaxed via homeomorphism.

Say that f is *convex-valued* iff for every $x \in X$, the set $f(x)$ is convex. Say that a correspondence f on X has a *closed graph* iff the set

$$\text{graph}(f) = \{(x, y) \in X^2 : y \in f(x)\}$$

is closed as a subset of X^2. Any continuous *function* f has a closed graph.

Theorem 7 (Kakutani Fixed Point Theorem)**.** *If $X \subseteq \mathbb{R}^N$ is non-empty, compact and convex, then every correspondence $f : X \to \mathbb{P}(X)$ that is non-empty-valued, convex-valued, and has a closed graph has a fixed point.*

Proof. For each $t \in \{1, 2, \ldots\}$, define the correspondence $\phi_t : X \to \mathbb{P}(X)$ by, for any $\hat{x} \in X$, $\hat{y} \in \phi_t(\hat{x})$ iff $\hat{y} \in X$ and there is a point (x, y) in the graph of f such that $d((x, y), (\hat{x}, \hat{y})) < 1/t$. The graph of ϕ_t looks like a tube around the graph of f. The correspondence ϕ_t is non-empty valued, convex-valued, and the set given by the graph of ϕ_t is open relative to $X \times X$. By Michael's Selection Theorem (on which, see Border (1985)), there is a continuous function $g_t : X \to X$ such that for each $x \in X$, $g_t(x) \in \phi_t(x)$. g_t is called a *continuous selection* from ϕ_t.

By the Brouwer fixed point theorem (and in particular Theorem 5), for each t, there is a $\hat{x}_t \in X$ such that $g_t(\hat{x}_t) = \hat{x}_t$. For each t, by construction of ϕ_t and g_t, there is an (x_t, y_t) in the graph of f such that $d((x_t, y_t), (\hat{x}_t, \hat{x}_t)) < 1/t$. Since $X \times X$ is compact, there is an $x^* \in X$ such that a subsequence of $(\hat{x}_t, \hat{x}_t)$ converges to (x^*, x^*). By the triangle inequality, along this same subsequence, (x_t, y_t) must likewise converge to (x^*, x^*). Since the graph of f is closed, (x^*, x^*) is in the graph of f, hence $x^* \in f(x^*)$, as was to be shown. ■

The examples of non-existence of a fixed point under Brouwer provide examples of non-existence under Kakutani when X is not compact or the graph of f is not closed (a function is a special case of a corresondence). The important new condition introduced by Kakutani is that f be convex-valued.

Example 9. Let $X = [0, 1]$ and let

$$f(x) = \begin{cases} 1 & \text{if } x < 1/2, \\ \{0, 1\} & \text{if } x = 1/2, \\ 0 & \text{if } x > 1/2. \end{cases}$$

There is no fixed point. Here, X is compact and f is non-empty valued and has a closed graph, but it is not convex valued.

Note that this example is similar to Example 2 in Section 2.3, but f here is a closed graph correspondence whereas f in Example 2 was a function that did not have a closed graph. □

Remark 1. Many treatments of Kakutani are stated in terms of f being upper hemi-continuous and closed valued rather than f having a closed graph. Since $f : X \to X$ and X is compact, the two formulations are equivalent. □

2.5 The Eilenberg-Montgomery Fixed Point Theorem.

The Eilenberg-Montgomery (EM) Fixed Point Theorem generalizes Theorem 3 in two ways.

- It weakens "X is homeomorphic to a compact, convex set" to "X is an acyclic absolute neighborhood retract." I will not give a formal definition of these terms; see, for example, McLennan (2008).

  An example of a compact set that satisfies the EM condition but that is not homeomorphic to a convex set is the "+" sign.

- EM holds in any metric space, including infinite-dimensional metric spaces.

A paper in which EM plays a critical role is Reny (2011).

Theorem 8 (The Eilenberg-Montgomery Fixed Point Theorem)**.** *If a non-empty metric space X is a compact acyclic absolute neighborhood retract then it has the fixed point property.*

Proof. I don't know of any easily accessible citations for a proof of this theorem. The original paper is Eilenberg and Montgomery (1946). A standard citation is Borsuk (1967). ■

Remark 2. Theorem 8 has a variant for correspondences that is analogous to the Kakutani Fixed Point Theorem (actually, the version for correspondences is the version originally stated). □

3 The Contraction Mapping Theorem

In this section and the next, I develop two additional fixed point theorems that are distinct from Brouwer and its relatives. This section focuses on the Contraction Mapping Theorem, which places only an extremely weak restriction on the domain but imposes a very strong continuity condition on f. The next, and last, section focuses on the Tarski Fixed Point Theorem, which states that f has a fixed point if it is weakly increasing, but not necessarily continuous, provided that the domain is, loosely, a generalized rectangle (possibly with holes). Both results hold in infinite dimensional spaces and both have proofs that are relatively easy.

To motivate the Contraction Mapping Theorem, consider first the case of an affine function on $\mathbb{R}$: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$ where $a, b \in \mathbb{R}$. If $a \neq 1$, then f has the fixed point

$$x^* = \frac{b}{1-a}.$$

(If $a = 1$ and $b = 0$ then every point is a fixed point. If $a = 1$ and $b \neq 0$ then there is no fixed point.) Note that the domain of f here is not compact.

The following provides an algorithm for finding x^*. Of course, I already have a formula for x^*, so I don't need an algorithm. But the algorithm generalizes, whereas the formula for x^* does not.

Suppose first that $|a| < 1$. Take x_0 to be any point in $\mathbb{R}$. Let $x_1 = f(x_0)$, $x_2 = f(x_1) = f(f(x_0)), \ldots$. Then $x_t \to x^*$. This is easiest to see if $b = 0$, in which case $x^* = 0$. Then $x_1 = ax_0$, $x_2 = a^2x_0$, $\ldots$, $x_t = a^t x_0$, $\ldots$. Since $|a| < 1$, $a^t \to 0$, which implies $x_t \to 0$, as was to be shown. If instead $|a| > 1$, then simply invert $y = ax + b$ to get,

$$x = f^{-1}(y) = \frac{1}{a}y - \frac{b}{a}.$$

Take y_0 to be any point in $\mathbb{R}$, $y_1 = f^{-1}(y_0)$, and so on. Since $|a| > 1$, $|1/a| < 1$, and hence $y_t \to y^* = x^*$.

The Contraction Mapping Theorem extends this argument to non-linear functions in arbitrary complete metric spaces. In the case where the domain is $\mathbb{R}$ and the function is differentiable, the analog to the requirement in the affine case that $|a| < 1$ is that there is a number $c \in [0, 1)$ such that for every $x \in \mathbb{R}$,

$$|Df(x)| \leq c < 1.$$

For any $x, \hat{x} \in \mathbb{R}$, the Mean Value Theorem says that there is an $x_m \in (x, \hat{x})$ such that

$$\frac{f(\hat{x}) - f(x)}{\hat{x} - x} = Df(x_m)$$

which implies

$$|f(\hat{x}) - f(x)| \leq c\,|\hat{x} - x|.$$

This motivates the following definition.

Definition 4. *Let (X, d) be a metric space. A function $f : X \to X$ is a* contraction *iff there is a number $c \in [0, 1)$ such that for any $\hat{x}, x \in X$,*

$$d(f(\hat{x}), f(x)) \leq c\, d(\hat{x}, x).$$

Theorem 9 (Contraction Mapping theorem)**.** *Let (X, d) be a non-empty complete metric space. Then any contraction $f : X \to X$ has a unique fixed point.*

Proof. Take any $x_0 \in X$ and form the sequence x_0, $x_1 = f(x_0)$, $x_2 = f(x_1) = f(f(x_0))$, and so on. It is an easy exercise to show that if f is a contraction then this sequence is Cauchy. Since X is complete, there is an $x^* \in X$ such that $x_t \to x^*$. Since $x_{t+1} = f(x_t)$, and $x_t \to x^*$, this implies that $f(x_t) \to x^*$. Since f is a contraction, it is (trivially) continuous. Hence $f(x^*) = x^*$, which establishes that x^* is a fixed point. Finally, if x^* and $\hat{x}$ are both fixed points then, since f is a contraction, $d(\hat{x}, x^*) \leq cd(\hat{x}, x^*)$, which implies that $d(\hat{x}, x^*) = 0$, which establishes that x^* is the unique fixed point. ■

Remark 3. If f is invertible and f^{-1} is a contraction then f^{-1} has a fixed point, and hence so does f. □

Example 10. Let $X = [0, 1)$ and $f(x) = (1 + x)/2$. There is no fixed point. Here, f is a contraction but X is not complete. □

Example 11. Let $X = \mathbb{R}$ and $f(x) = x + 1/2$. There is no fixed point. Here X is complete but f is not a contraction. Moreover, although f is invertible, f^{-1} is not a contraction either. □

4 The Tarski Fixed Point Theorem

In Example 2 in Section 2.3, a fixed point fails to exist because f is not continuous. On closer inspection, the problem is that, at the discontinuity, f jumps down. On the domain $[0, 1]$, one can show that a fixed point must exist if f is weakly increasing, even if f is highly discontinuous. The Tarski Fixed Point Theorem extends this intuition to much more general domains.

Let X be a partially ordered set. The canonical example is $\mathbb{R}^N$, where $\hat{x} \geq x$ iff $\hat{x}_n \geq x_n$ for all n. Thus $(3, 3) \geq (2, 2)$ but neither $(1, 0) \geq (0, 1)$ nor $(1, 0) \leq (0, 1)$.

Given a set $S \subseteq X$, an upper bound of S in X is an element $x \in X$ such that $x \geq s$ for all $s \in S$. Say that x is the *least* upper bound of S in X iff (i) x is an upper bound of S in X and (ii) if $\hat{x}$ is an upper bound of S in X then $\hat{x} \geq x$. The least upper bound, if it exists, is unique since if x and $\hat{x}$ are both least upper bounds then $x \geq \hat{x}$ and $\hat{x} \geq x$. Lower bounds and greatest lower bounds are defined analogously.

Definition 5. *A partially ordered set X is a* lattice *iff every $A \subseteq X$ consisting of exactly two elements has a least upper bound and a greatest lower bound in X.*

$X = \mathbb{R}^N$ is a standard example of a lattice.

Example 12. Let,

$$\begin{aligned} X &= \{(0, 0), (0, 1), (1, 0), (1, 1)\}, \\ \hat{X} &= \{(0, 0), (1, 0), (0, 1), (2, 2)\}, \\ \tilde{X} &= \{(0, 0), (1, 0), (0, 1)\}. \end{aligned}$$

Both X and $\hat{X}$ are lattices. In particular if $S = \{(0,1),(1,0)\}$, then the least upper bound of S in X is $(1,1)$ while the least upper bound of S in $\hat{X}$ is $(2,2)$. On the other hand, $\tilde{X}$ is not a lattice. □

Remark 4. In the previous example, notice that $\hat{X}$ is a lattice even though the least upper bound of $S = \{(0,1),(1,0)\}$ in $\mathbb{R}^2$, namely $(1,1)$, is not an element of $\hat{X}$. If Z is a lattice and $X \subseteq Z$, then X is a *sublattice* of Z iff for any set $S \subseteq X$ consisting of two elements, the least upper bound and greatest lower bound of S in Z are also in X. Thus, in the example above, X is a sublattice of $\mathbb{R}^2$ while $\hat{X}$ is not. Sublattices are important in some applications but they do not play a role in the Tarski Fixed Point Theorem. □

Definition 6. *A partially ordered set X is a* complete lattice *iff every $A \subseteq X$ has a least upper bound and greatest lower bound in X.*

A complete lattice is automatically a lattice. Notice that a complete lattice must be bounded (take $A = X$). A complete lattice need not be complete in the metric space sense (e.g., every Cauchy sequence converges) even when X is a metric space.

Example 13. The set $X = [0, 1/2) \cup \{1\}$ is a complete lattice. In particular, the least upper bound of $S = [0, 1/2)$ is 1. But X is not complete in the metric space sense. □

Given a partially ordered set X, an *interval* in X is a set of the form $\{x \in X : a \leq x \leq b\}$, where $a, b \in X$, $a \leq b$. In $X = \mathbb{R}^N$, an interval is a rectangle (or multidimensional analog thereof), with sides parallel to the axes. In Example 13, if $R = X$ then R is (trivially) an interval in X even though it is not an interval in $\mathbb{R}$.

Intervals are important in part because intervals in complete lattices are themselves complete lattices.

Theorem 10. *If X is complete lattice, then for any $a, b \in X$ with $a \leq b$, the interval $\{x \in X : a \leq x \leq b\}$ is a complete lattice.*

Proof. Let $a, b \in X$, $a \leq b$, and let $R = \{x \in X : a \leq x \leq b\}$ (R for "rectangle"). Consider any $S \subseteq R$. Since X is complete, S has a least upper bound $s^* \in X$. I will show that $s^* \in R$. Since b is an upper bound for R, b is an upper bound for S, and hence $s^* \leq b$. Since a is a lower bound for R, a is a lower bound for S, and hence for any $x \in S$, $a \leq x \leq s^*$. Thus $a \leq s^* \leq b$, hence $s^* \in R$. And similarly, the greatest lower bound of S is in R. Therefore, any $S \subseteq R$ has a greatest lower bound and a least upper bound in R, and hence R is a complete lattice. ■

If X is not complete then an interval in X need not be complete.

Example 14. Suppose that $X = [0, 1/2) \cup (3/4, 1]$. Then $R = X$ is an interval in X that is not complete, since $S = [0, 1/2)$ has no least upper bound in R. □

Finally, a function $f : X \to X$ is *weakly increasing* iff $\hat{x} \geq x$ implies $f(\hat{x}) \geq f(x)$. Again, since X is only partially ordered, there may be many pairs of x and $\hat{x}$ for which this property has no bite.

Theorem 11 (Tarski Fixed Point Theorem)**.** *Let X be a non-empty complete lattice. If $f : X \to X$ is weakly increasing, then the set of fixed points of f is a non-empty complete lattice.*

Before proving the theorem, let me make two remarks. First, the condition that X is a lattice is extremely general and encompasses many (perhaps most, maybe even all) settings of economic interest.

Second, as already noted, the conclusion that the set of fixed points, call it P, is a complete lattice implies that P has a largest element b^* and smallest element a^*. As an example of what P might look like, consider the following.

Example 15. Let $X = [0, 1]$. Let

$$f = \begin{cases} x & \text{if } x < 1/2, \\ 1 & \text{if } x \geq 1/2 \end{cases}$$

Then $P = [0, 1/2) \cup \{1\}$. A subtlety here is that P is a complete lattice even though it is *not* complete when viewed as a *sub*lattice of $[0, 1]$. □

Thus, if $P \subseteq \mathbb{R}$ then P, even though complete in the lattice theoretic sense, need not be closed. But the fact that P is complete implies that it must have a largest (and smallest) element. This rules out the possibility that, for example, $P = [0, 1/2)$.

Proof of the Tarski Fixed Point Theorem.

Let P denote the set of fixed points. The proof is in two steps.

1. I first show that P is non-empty and has a largest element b^* and a smallest element a^*. For many economic applications, this is actually all of Tarski that is actually used.

   Since X is a complete lattice, it has a smallest element, a_0 and a largest element b_0. Let
   $$A = \{x \in X : x \leq f(x)\}.$$
   $a_0 \in A$, hence A is not empty. Since X is complete, A has a least upper bound in X. Call this least upper bound b^*. I will show that b^* is a fixed point.

   I claim first that $b^* \in A$. To see this, note that for any $x \in A$, since $x \leq b^*$ and f is weakly increasing, $f(x) \leq f(b^*)$. Moreover, since $x \in A$, $x \leq f(x)$. Combining inequalities, $x \leq f(b^*)$. Since this holds for every $x \in A$, this establishes that $f(b^*)$ is an upper bound of A. Since b^* is the least upper bound,
   $$b^* \leq f(b^*),$$
   which means that b^* satisfies the condition for inclusion in A.

I next claim that $f(x) \in A$. To see this, note that by definition, if $x \in A$ then $x \leq f(x)$. Since f is weakly increasing , $f(x) \leq f(f(x))$, which is the condition for $f(x) \in A$, which establishes the claim.

This implies, in particular, that since $b^* \in A$, $f(b^*) \in A$. Since b^* is an upper bound for A, this means that

$$f(b^*) \leq b^*.$$

Combining the inequalities in b^* and $f(b^*)$ yields $f(b^*) = b^*$: b^* is a fixed point.

Moreover, $P \subseteq A$ and hence, since b^* is an upper bound of A, it is an upper bound of P: b^* is the largest fixed point.

A similar argument establishes that the greatest lower bound of $B = \{x \in X : x \geq f(x)\}$ is also a fixed point. Explicitly, let a^* be the greatest lower bound of B. For any $x \in B$, $a^* \leq x$, hence (since f is weakly increasing), $f(a^*) \leq f(x)$. Since $x \in B$, $f(x) \leq x$, hence $f(a^*) \leq x$. Since this holds for any $b \in B$, $f(a^*)$ is a lower bound of B. Since a^* is the greatest lower bound, $a^* \geq f(a^*)$, which implies $a^* \in B$. Moreover, since $a^* \geq f(a^*)$ and f is weakly increasing, $f(a^*) \geq f(f(a^*)$, which implies $f(a^*) \in B$. Since a^* is a lower bound of B, $a^* \leq f(a^*)$. Putting all this together, $a^* = f(a^*)$, hence $a^* \in B$. Since $P \subseteq B$ and a^* is a lower bound of B, a^* is a lower bound of P: a^* is the smallest fixed point.

2. It remains to show that P is, in fact, a complete lattice. Let S be any non-empty subset of P. I need to show that S has a least upper bound and a greatest lower bound in P.

   Since X is complete, S has a least upper bound $s^* \in X$. Since $S \subseteq P$, this implies $s^* \leq b^*$. Let $R = \{x \in X : s^* \leq x \leq b^*\}$. By Theorem 10, R is a complete lattice. I show below that f maps R into itself. Therefore, it follows by the first step that f has a smallest fixed point in R, call it r^*. I claim that r^* is the least upper bound of S in P. It is an upper bound since it is in R. It is the least upper bound of S in P because it is less than or equal to any other fixed point in R, and hence less than or equal to any fixed point that is an upper bound of S. If s^* happens to be a fixed point then $r^* = s^*$. In Example 15, if $S = [0, 1/2)$ then $s^* = 1/2$ while $r^* = 1$, so $s^* < r^*$.

   It remains to show that f maps R into itself. Consider any $x \in R$. Since f is weakly increasing, $x \leq b^*$, and b^* is a fixed point, $f(x) \leq f(b^*) = b^*$. Similarly, if $x \in R$ then $x \geq s^*$, hence $x \geq s$ for all $s \in S$. Therefore, since f is weakly increasing and every $s \in S$ is a fixed point, $f(x) \geq f(s) = s$ for all $s \in S$. Hence $f(x)$ is an upper bound of S. Since s^* is the least upper bound, $f(x) \geq s^*$. Thus, $s^* \leq f(x) \leq b^*$, hence $f(x) \in R$, as was to be shown.

The proof that S has a greatest lower bound in P is analogous and I omit it. Thus, S has least upper bound and greatest lower bound in P. Since S was arbitrary, it follows that P is a complete lattice.

■

Example 16. Let $X = [0,1]$ and let

$$f(x) = \begin{cases} 1 - x/2 & \text{if } x \le 1/2, \\ 1/2 - x/2 & \text{if } x > 1/2. \end{cases}$$

There is no fixed point. In this case, X is a complete lattice but f is not weakly increasing. □

Example 17. As in Example 1, let $X = [0,1)$ and $f(x) = (x+1)/2$. There is no fixed point. Here, f is increasing but X is not a complete lattice. □

As noted in the proof of Theorem 11, since X is a complete lattice it has a largest element b_0 and a smallest element a_0. Let $a_1 = f(a_0)$, $a_2 = f(a_1) = f(f(a_0))$, and so on. Similarly, let $b_1 = f(b_0)$, $b_1 = f(f(b_0))$, and so on. Since f maps X into itself, it must be that $a_1 \ge a_0$ and $b_1 \le b_0$. Moreover, since $a_0 \le b_0$ and f is weakly increasing, $a_1 \le b_1$. In addition, since $a_0 \le a_1$ and f is weakly increasing, $f(a_0) \le f(a_1)$. Moreover, since $a^* \ge a_0$, a^* is a fixed point, and f is weakly increasing, $a_1 = f(a_0) \le f(a^*) = a^*$. By induction, it follows that $a_t \le a^*$ for all t. By an analogous argument, $b_t \ge b^*$ for all t. Therefore,

$$a_0 \le a_1 \le \cdots a^* \le b^* \cdots \le b_1 \le b_0.$$

If X is finite then it is easy to see that $a_t \to a^*$ in the trivial sense that there is a T such that $a_t = a^*$ for all $t \ge T$. Similarly $b_t \to b^*$. If X is not finite then the situation is more delicate.

Example 18. Let $X = [0,1/4) \cup [3/8,1/2) \cup \{3/4\}$ and let

$$f(x) = \begin{cases} x/2 + 1/8 & \text{if } x \in [0,1/4), \\ x/2 + 4/16 & \text{if } x \in [3/8,1/2), \\ 3/4 & \text{if } x = 3/4. \end{cases}$$

Then $a^* = b^* = 3/4$. Considered as a subset of $\mathbb{R}$, $a_t \to 1/4 \ne a^*$. Moreover, the least upper bound of $\{a_0, a_1, \dots\}$ within X is $3/8$, which is not a fixed point. So there is no sense in which a_t converges to a fixed point. □

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