Game Theory Basics I:
Strategic Form Games

1 Introduction.

Game theory is a mathematical framework for analyzing conflict and cooperation. Early work was motivated by gambling and recreational games such as chess, hence the “game” in game theory. But it quickly became clear that the framework had much broader application. Today, game theory is used for mathematical modeling in a wide range of disciplines, including many of the social sciences, computer science, and evolutionary biology. Here, I draw examples mainly from economics.

These notes are an introduction to a mathematical formalism called a strategic form game (also called a normal form game). For the moment, think of a strategic form game as representing an atemporal interaction: each player (in the language of game theory) acts without knowing what the other players have done. An example is a single instance of the two-player game Rock-Paper-Scissors (probably already familiar to you, but discussed in the next section).

In companion notes, Game Theory Basics II: Extensive Form Games, I develop an alternative formalism called an extensive form game. Extensive form games explicitly capture temporal considerations, such as the fact that in standard chess, players move in sequence, and each player knows the prior moves in the game. As I discuss in the notes on extensive form games, there is a natural way to give any extensive form game a strategic form representation.

There is a third formalism called a game in coalition form (also called characteristic function form). The coalition form abstracts away from the details of what individual players do and focuses instead on what payoff allocations are physically possible, both for all players taken together and for every subset (coalition) of players. I do not (yet) have notes on games in coalition form.

A nice, short introduction to the study of strategic and extensive form games is Osborne (2008). A standard undergraduate text on game theory is Gibbons (1992). Watson (2013) and Tadelis (2013) are also good. A standard graduate game theory text is Fudenberg and Tirole (1991); I also like Osborne and Rubinstein (1994). There are also good introductions to game theory in graduate microeconomic theory texts such as Kreps (1990), Mas-Colell, Whinston and Green (1995), and Jehle and

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1 This work is licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 4.0 License.
Reny (2000). Finally, Luce and Raiffa (1957) is a classic and still valuable text on game theory, especially for discussions of interpretation and motivation.

I start my discussion of strategic form games with Rock-Paper-Scissors.

2 An example: Rock-Paper-Scissors.

The game Rock-Paper-Scissors (RPS) is represented in Figure 1 in what is called a game box. There are two players, 1 and 2. Each player has three strategies in the game: R (rock), P (paper), and S (scissors). Player 1 is represented by the rows while player 2 is represented by the columns. If player 1 chooses R and player 2 chooses P then this is represented as the pair, called a strategy profile, (R,P); for this profile, player 1 gets a payoff of -1 and player 2 gets a payoff of +1, represented as a payoff profile (-1,1).

For interpretation, think of payoffs as encoding preferences over winning, losing, or tying, with the understanding that S beats P (because scissors cut paper), P beats R (because paper can wrap a rock . . . ), and R beats S (because a rock can smash scissors). If both choose the same, then they tie. The interpretation of payoffs is actually quite delicate and I discuss this issue at length in Section 3.3.

This game is called zero-sum because, for any strategy profile, the sum of payoffs is zero. In any zero-sum game, there is a number V, called the value of the game, with the property that player 1 can guarantee that she gets at least V in expectation no matter what player 2 does and conversely player 2 can get −V no matter what player 1 does. I provide a proof of this theorem in Section 4.5. In this particular game, V = 0; either player can guarantee that they get 0 in expectation by randomizing evenly over the three strategies.

Note that randomization is necessary to guarantee an expected payoff of at least 0. In Season 4 Episode 16 of the Simpsons, Bart persistently plays Rock against Lisa, and Lisa plays Paper, and wins. Bart here doesn’t even seem to understand the game box, since he says, “Good old Rock. Nothing beats that.” I discuss the interpretation of randomization in Section 3.4.
3 Strategic Form Games.

3.1 The Strategic Form.

I restrict attention in the formalism to finite games: finite numbers of players and finite numbers of strategies. Some of the examples, however, involve games with infinite numbers of strategies.

A strategic form game is a tuple \((I, (S_i)_i, (u_i)_i)\).

- \(I\) is the finite set, assumed not empty, of players with typical element \(i\). The cardinality of \(I\) is \(N\); I sometimes refer to \(N\)-player games. To avoid triviality, assume \(N \geq 2\) unless explicitly stated otherwise.

- \(S_i\) is the set, assumed finite and not empty, of player \(i\)'s strategies, often called pure strategies to distinguish from the mixed strategies described below. \(S = \prod S_i\) is the set of pure strategy profiles, with typical element \(s = (s_1, \ldots, s_N) \in S\).

- \(u_i\) is player \(i\)'s payoff function: \(u_i: S \to \mathbb{R}\). I discuss the interpretation of payoffs in Section 3.3.

Games with two players and small strategy sets can be represented via a game box, as in Figure 1 for Rock-Paper-Scissors. In that example, \(I = \{1, 2\}, S_1 = S_2 = \{R, P, S\}\), and payoffs are as given in the game box.

As anticipated by the RPS example, we will be interested in randomization. For each \(i\), let \(\Sigma_i\) be the set of probabilities over \(S_i\), also denoted \(\Delta(S_i)\). An element \(\sigma_i \in \Sigma_i\) is called a mixed strategy for player \(i\). Under \(\sigma_i\), the probability that \(i\) plays pure strategy \(s_i\) is \(\sigma_i[s_i]\).

A pure strategy \(s_i\) is equivalent to a degenerate mixed strategy, with \(\sigma_i[s_i] = 1\) and \(\sigma_i[\hat{s}_i] = 0\) for all \(\hat{s}_i \neq s_i\). Abusing notation, I use the notation \(s_i\) for both the pure strategy \(s_i\) and for the equivalent mixed strategy.

Assuming that the cardinality of \(S_i\) is at least 2, the strategy \(\sigma_i\) is fully mixed iff \(\sigma_i[s_i] > 0\) for every \(s_i \in S_i\). \(\sigma_i\) is partly mixed iff it is neither fully mixed nor pure (degenerate).

Since the strategy set is assumed finite, I can represent a \(\sigma_i\) as a vector in either \(\mathbb{R}^{|S_i|}\) or \(\mathbb{R}^{|S_i|}-1\), where \(|S_i|\) is the number of elements in \(S_i\). For example, if \(S_1\) has two elements, then I can represent \(\sigma_1\) as either \((p, 1-p)\), with \(p \in [0, 1]\) (probability \(p\) on the first strategy, probability \(1-p\) on the second), or just as \(p \in [0, 1]\) (with the probability on the second strategy inferred to be \(1-p\)). And a similar construction works for any finite strategy set. Note that under either representation, \(\Sigma_i\) is compact and convex.

I discuss the interpretation of mixed strategies in Section 3.4. For the moment, however, suppose that players might actually randomize, perhaps by making use of coin flip or toss of a die. In this case the true set of strategies is actually \(\Sigma_i\) rather
than $S_i$. One can think of $S_i$ as a concise way to represent the true strategy set, which is $\Sigma_i$.

$\Sigma = \prod_i \Sigma_i$ is the set of mixed strategy profiles, with typical element $\sigma = (\sigma_1, \ldots, \sigma_N)$. A mixed strategy profile $\sigma$ induces an independent probability distribution over $S$.

**Example 1.** Consider a two-player game in which $S_1 = \{T, B\}$ and $S_2 = \{L, R\}$. If $\sigma_1[T] = 1/4$ and $\sigma_2[L] = 1/3$ then the induced distribution over $S$ can be represented in a game box as in Figure 2. □

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**Figure 2:** An independent distribution over strategy profiles.

Abusing notation, let $u_i(\sigma)$ be the expected payoff under this independent distribution; that is,

$$u_i(\sigma) = \mathbb{E}_{\sigma}[u_i(s)] = \sum_{s \in S} u_i(s)\sigma_1[s_1]\ldots\sigma_N[s_N].$$

where $\mathbb{E}_\sigma$ is the expectation with respect to the probability distribution over $S$ induced by $\sigma$.

**Example 2.** In RPS (Figure 1), if Player 1 plays $R$ and Player 2 randomizes $(1/6, 1/3, 1/2)$, then Player 1’s expected payoff is

$$0 \times 1/6 + (-1) \times 1/3 + 1 \times 1/2 = 1/6.$$

□

Finally, I frequently use the following notation. A strategy profile $s = (s_1, \ldots, s_N)$ can also be represented as $s = (s_i, s_{-i})$, where $s_{-i} \in \prod_{j \neq i} S_j$ is a profile of pure strategies for players other than $i$. Similarly, $\sigma = (\sigma_i, \sigma_{-i})$ is alternative notation for $\sigma = (\sigma_1, \ldots, \sigma_N)$, where $\sigma_{-i} \in \prod_{j \neq i} \Sigma_j$ is a profile of mixed strategies for players other than $i$.

### 3.2 Correlation.

The notation so far builds in an assumption that any randomization is independent. To see what is at issue, consider the following game.

**Example 3.** The game box for one version of Battle of the Sexes is in Figure 3. The players would like to coordinate on either $(A, A)$ or $(B, B)$, but they disagree about which of these is better. □
A game box for Battle of the Sexes.

If players in the Battle of the Sexes game depicted in Figure 3 can coordinate (say by pre-play communication) then it is possible that they would toss a coin prior to play and then execute \((A, A)\) if the coin lands heads and \((B, B)\) if the coin lands tails. This induces a correlated distribution over strategy profiles, which I represent in Figure 4.

![Figure 4: A correlated distribution over strategy profiles for Battle of the Sexes.](image)

Note that under this correlated distribution, each player plays \(A\) half the time. If players were instead to play \(A\) half the time independently, then the distribution over strategy profiles would be as in Figure 5.

![Figure 5: An independent distribution over strategy profiles for Battle of the Sexes.](image)

The space of all probability distributions over \(S\) is \(\Sigma^c\), also denoted \(\Delta(S)\), with generic element denoted \(\sigma^c\). Abusing notation (again), I let \(u_i(\sigma^c) = \mathbb{E}_{\sigma^c}[u_i(s)] = \sum_{s \in S} u_i(s)\sigma^c[s]\). For strategies for players other than \(i\), the notation is \(\sigma^c_{-i} \in \Sigma^c_{-i} = \Delta(S_{-i})\). The notation \((\sigma_i, \sigma^c_{-i})\) denotes the distribution over \(S\) for which the probability of \(s = (s_i, s_{-i})\) is \(\sigma_i[s_i]\sigma^c_{-i}[s_{-i}]\). Any element of \(\Sigma\) induces an element of \(\Sigma^c\): an independent distribution over \(S\) is a special form of correlated distribution over \(S\).

Remark 1. A mixed strategy profile \(\sigma \in \Sigma\) is not an element of \(\Sigma^c\), hence \(\Sigma \not\subseteq \Sigma^c\). Rather, \(\sigma\) induces a distribution over \(S\), and this induced distribution is an element of \(\Sigma^c\).

In the Battle of the Sexes of Figure 3, to take one example, it takes three numbers to represent an arbitrary element \(\sigma^c \in \Sigma^c\) (only three, rather than four, because the four numbers have to add up to 1, hence one number can always be inferred from the other three). In contrast, each mixed strategy \(\sigma_i\) can be described by a single
number, as discussed earlier, hence $\sigma$ can be represented as a pair of numbers, which induces a distribution over $S$. Thus, in this example, the set of independent strategy distributions over $S$ is two dimensional while the set of all (correlated) strategy distributions is three dimensional. More generally, the set of independent strategy distributions over $S$ is a lower dimensional subset of $\Sigma^c$. $\square$

3.3 Interpreting Payoffs.

In most applications in economics, payoffs are intended to encode choice by decision makers, and this brings with it subtleties that affect the interpretation and testability of the theory. The material in this section assumes some familiarity with standard decision theory, although I try to make the discussion as self-contained as possible.

To make the issues more transparent, I first introduce some additional structure. Let $X$ be a finite set of possible outcomes for the game. Let $\gamma(s)$ give the outcome associated with strategy profile $s$; thus $\gamma : S \rightarrow X$.

- An outcome can encode (or represent) a profile of “prizes” for the players. In some games, this may take the form of a profile of numbers, which can be interpreted as monetary payments or penalties. This is frequently the case in economics applications of game theory. But the “prizes” can also be more abstract. For example, an outcome can specify which player (or players) “won” (without any more tangible prize), as in many recreational games. An outcome can be an assignment of partners (romantic, professional, organ donor). It can be multidimensional, as in market games where players trade bundles (vectors) of commodities. The possibilities are endless. Prizes can even be probabilistic; a prize might be a lottery ticket, for example.

- In addition to encoding prizes, an outcome can also encode how the game was played. For example, as discussed in the companion notes (Game Theory Basics II: Extensive Form Games), chess can be represented as a game in strategic form. For chess, an outcome might encode not only who won (the prize) but also a record of the exact sequence of moves by the players.

Example 4. Recall RPS (Rock-Paper-Scissors) from Section 2. By assumption, the possible outcomes are

$$\{(\text{win, lose}), (\text{lose, win}), (\text{tie, tie})\},$$

where, for example, $(\text{win, lose})$ means that player 1 gets “win” (whatever that might mean) while player 2 gets “lose.” In particular, I assume that the players have no particular interest in what strategies are played per se, only in who wins. $\gamma$ is given by, for example, $\gamma(R, P) = (\text{lose, win})$. $\square$

I assume that player choice can be represented by a preference relation that is a complete preorder over $X$. That is, there is a complete preorder, which I call
preferences, such that, given any subset of \( X \), an outcome \( x \) is the most highly ranked outcome in that subset if, in fact, the player would have chosen \( x \) from that subset. This decision theoretic assumption is substantive (it builds in transitivity, in particular) but it is implicit in all of standard game theory. Moreover (and, because I have assumed that \( X \) is finite, this is a theorem rather than a new assumption), player \( i \)’s preferences can in turn be represented by a utility function, say \( v_i : X \to \mathbb{R} \), where \( v_i(x) > v_i(\hat{x}) \) iff player \( i \) strictly prefers \( x \) to \( \hat{x} \). \( v_i \) will typically not be unique: two different utility functions represent the same preferences over \( X \) iff either is a strictly increasing function of the other.

**Example 5.** Returning to RPS, to simplify the discussion first note that the possible outcomes are exactly identified by which player wins or if they tie.

Suppose that either player strictly prefers win to tie and strictly prefers tie to lose. Any utility function that assigns a higher number to win than tie, and a higher number to tie than lose, will work. For example, I could have that, for player 1, win is -9, tie is -10, and lose is -100 and for player 2 that win is 11008, tie is 1, and lose is -90. In fact, and this is arbitrary at this point in the discussion, I take the utilities for both players to be win is 1, tie is 0, and lose is -1. □

Thus far, I have focused on pure strategy profiles. We are, however, also interested in mixed strategy profiles, which will induce probability distributions over \( X \). The assumption in standard game theory is that players are expected utility maximizers. Formally, this means the following. As was the case for \( X \), it is assumed that player choice over \( \Delta(X) \) can be represented by a complete preorder over \( X \) and that this preorder can, in turn, by represented by a utility function (because \( \Delta(X) \) is uncountably infinite, the second of these steps now requires an auxiliary technical assumption). As before, the utility function representing choice/preferences over \( \Delta(X) \) is not unique. Expected utility maximization introduces the new assumption that out of the set of possible utility functions that represent \( i \)’s preferences over \( \Delta(X) \), there is one, say \( V_i : \Delta(X) \to \mathbb{R} \), for which there is a utility function over \( X \), say \( v_i : X \to \mathbb{R} \), such that, for any probability \( \lambda \in \Delta(X) \), \( V_i(\lambda) \) is the expectation, with respect to \( \lambda \), of \( v_i(x) \):

\[
V_i(\lambda) = \mathbb{E}_\lambda[v_i(x)].
\]

If \( \lambda_x \) is the degenerate distribution that assigns probability 1 to outcome \( x \) then \( V_i(\lambda_x) = v_i(x) \).

\( v_i \) is potentially much more tightly pinned down than before; it not only has to rank outcomes in \( X \) correctly but it has to have the additional property that expectations of \( v_i \) rank probabilities in \( \Delta(X) \) correctly. It is possible for two different \( v_i \) to rank outcomes in \( X \) the same way but rank probability distributions in \( \Delta(X) \) differently. I will come back to this issue both immediately below and in Section 4.1, which discusses an explicit example.

Finally, the payoff function is simply the composition of the \( v_i \) and \( \gamma \) functions: \( u_i(s) = v_i(\gamma(s)) \). Similarly (and abusing notation, as usual), \( u_i(\sigma) = \mathbb{E}_\sigma[v_i(\gamma(s))] \).
Example 6. Returning yet again to RPS, if, for example, \(v_1(\text{win, lose}) = 1\) then \(u_1(R, S) = 1\), and so on. Again, note that there is an implicit assumption that \(v_i\) is compatible with a \(V_i\) that represents the player’s choice over \(\Delta(X)\).

As an illustrative calculation, suppose that player 2 randomizes \((1/6, 1/3, 1/2)\). Then player 1’s expected payoffs are \(1/6\) from \(R\), \(-1/3\) from \(P\) and \(1/6\) from \(S\). So, for this particular mixture by player 2, player 1 is indifferent between \(R\) and \(S\) but prefers either to \(P\). □

With this foundation, I can now make a number of remarks about the interpretation of payoffs in decision theoretic terms.

1. Because the payoff function encodes choice, by construction, payoff maximization over \(X\) is built into payoffs; it is not a separate behavioral assumption. If you think players would make different choices, then you need to change the payoff representations to reflect this. And if you balk at the idea of using utility functions to represent choice, then we are going to have to move to some other formalism entirely; you can’t both reject utility functions and continue to use payoff functions.

2. There is an old and still lively debate about whether the expected utility assumption is a good one, both for game theory in particular and for economics more generally. See, for example, the discussion of expected utility in Kreps (2012).

3. Since we care about choice over \(\Delta(X)\), there is less flexibility in choosing \(v_i\) than there would be if we care only about representing choice over just \(X\). For general expected utility, two \(v_i\) represent the same choice over \(\Delta(X)\) iff either is an affine transformation of the other, with strictly positive slope: that is, \(\hat{v}_i = av_i + b\), with \(a > 0\). In fact, because the expected utility calculations that appear in game theory are special (they are constrained by the structure of the game), there is somewhat more flexibility in choosing \(v_i\) than this suggests. In a two-player game, for example, we can multiply all of player i’s payoffs by a positive constant (as above) and we can add a constant to all of player i’s payoffs corresponding to each pure strategy of the other player, with (potentially) a different constant for each of the other player’s pure strategies (this is the new part), without changing the player’s ranking over their mixed strategies. See Section 4.1 for more on this.

4. In many economic applications, an outcome is a profile of monetary values (profits, for example). It is common practice in such applications to assume that \(u_i(s)\) equals the prize to \(i\) if \(s\) gives \(i\) profits of $1 billion, then \(u_i(s) = \$1\) billion. This assumption is substantive.

   (a) The assumption rules out phenomena such as altruism or envy. In contrast, the general formalism allows, for example, for \(u_i(s)\) to be the sum of the individual prizes (a form of perfect altruism).
(b) The assumption rules out risk aversion. If players maximize expected payoffs, then to capture risk aversion, payoffs must be a concave function of profit. See also the example in Section 4.1.

(c) The assumption rules out the possibility that players care about more than the monetary prize, that they also care about how the game was played. The players might also care about environmental impact or worker welfare or national security implications, and so on. Again, these extra consideration are not ruled out by the game theory formalism itself but only by this particular implementation of that formalism.

5. It can be difficult to test game theory predictions such as Nash equilibrium in a lab. The experimenter can control $\gamma$, but the $v_i$, and hence the $u_i$, are in the heads of the subjects and not directly observable. In particular, it is not a violation of game theory to find that players are altruistic or spiteful. This flexibility of the game theory formalism is a feature, not a bug: the goal is to have a formalism that can be used to model essentially any strategic interaction.

6. A related point is that while it may be reasonable to assume that there is common knowledge of $\gamma$ (everyone knows the correct $\gamma$, everyone knows that everyone knows the correct $\gamma$, and so on), there may not be even mutual knowledge of the $v_i$ and hence of the $u_i$ (players may not know the utility functions of the other players). In particular, in economics contexts, even if it makes sense to assume that players have common knowledge of rankings over outcomes (e.g., even if it is common knowledge that everyone cares only about their own income, and that everyone prefers more income to less), $v_i$ may still not be common knowledge (because players may not know each other’s attitudes toward risk, for example).

Games in which there is not common knowledge of the $u_i$ are called games of incomplete information. I discuss approaches to modeling such environments later in the course.

The interpretation of payoffs in terms of decision theory is not the only one possible. For example, in some applications of game theory to evolutionary biology, the strategies might be alleles (alternative versions of a gene) and payoffs might be the expected number of offspring.

Remark 2. $V_i$ are $v_i$ are both often called Von Neumann-Morgenstern (VN-M) expected utility functions; usage varies. $v_i$ is also often called a Bernoulli utility function, especially for the special case in which the outcome is a profile of monetary payments (or penalties) and players care only about their own individual payments. I sometimes refer to $v_i$ as a subutility function.

The basic idea of expected utility is usually credited to Daniel Bernoulli (1700-1782), who focused on the special case of probabilities over monetary payments. von
Neumann and Morgenstern (1947), as part of their work on providing a foundation for game theory, extended expected utility to abstract outcomes and also provided an axiomatic foundation. There were, subsequently, other axiomatizations, of which the best known is Herstein and Milnor (1953). Savage (1954) extended the expected utility idea to frameworks in which probabilities are subjective. Anscombe and Aumann (1963) provided an alternate, and in some ways simpler, formulation of Savage’s theory for settings in which there is objective as well as subjective probability. (The leading example in Anscombe and Aumann (1963) is that the probabilities on a roulette wheel are objective while those for betting on horse races are subjective. A hardcore subjectivist would counter that all probabilities are subjective, even those for a roulette wheel or the toss of a coin or a die.) For more on decision theory in general and decision over probabilities in particular, see Kreps (1988), Mas-Colell, Whinston and Green (1995), and Kreps (2012).

3.4 Interpreting Randomization.

There are three main interpretations of randomization in games. These interpretations are not mutually exclusive.

1. **Objective Randomization.** Each player has access to a randomization device. In this case, the true strategy set for player \( i \) is \( \Sigma_i \). \( S_i \) is just a concise way to communicate \( \Sigma_i \).

   An interesting case of this is reported in Sontag and Drew (1998). Military submarines occasionally implement hard turns in order to detect possible trailing submarines; such maneuvers are called “clearing the baffles.” In order to be as effective as possible, it is important that these turns be unpredictable. Sontag and Drew (1998) reported that a captain of the USS Lapon used dice in order to randomize. Curiously, in Clancy (1984), a classic military techno-thriller, a critical plot point involves a CIA analyst correctly predicting when and how a (fictional) top Russian submarine commander would clear the baffles of his submarine.

2. **Empirical Randomization.** From the perspective of an observer (say an experimentalist), \( \sigma_i[s_i] \) is the frequency with which \( s_i \) is played.

   The observer could, for example, be seeing data from a cross section of play by different players (think of a Rock-Paper-Scissors tournament with many simultaneous matchings). \( \sigma_i[s_i] \) is the fraction of players in role \( i \) of the game who play \( s_i \). Nash discussed this interpretation explicitly in his thesis, Nash (1950b). Alternatively, the observer could be seeing data from a time series: the same players play the same game over and over, and \( \sigma_i[s_i] \) is the frequency with which \( s_i \) is played over time.

3. **Subjective Randomization.** From the perspective of player \( j \neq i \), \( \sigma_i[s_i] \) is the probability that \( j \) assigns to player \( i \) playing \( s_i \).
Consider again the cross sectional interpretation of randomization, in which many instances of the game are played by different players. If players are matched randomly and anonymously to play the game then, from the perspective of an individual player, the opponents are drawn randomly, and hence opposing play can be “as if” random even if the player knows that individual opponents are playing pure strategies.

An important variant of the cross sectional interpretation of randomization is the following idea, due to Harsanyi (1973). As discussed in Section 3.3, players may know the $\gamma$ of the game (giving prizes), but not the $v_i$ of the other players (giving preferences over prizes), and hence may not know the $u_i$ (giving payoffs) of the other players. Suppose that player $j$ assigns a probability distribution over possible $u_i$, and for each $u_i$, player $j$ forecasts play of some pure strategy $s_i$. Then, even though player $j$ thinks that player $i$ will play a pure strategy, player $i$’s play is effectively random in the mind of player $j$ because the distribution over $u_i$ induces a distribution over $s_i$.

A distinct idea is that in many games it can be important not to be predictable. This was the case in Rock-Paper-Scissors, for example (Section 2). A particular concern is that if the game is played repeatedly then one’s behavior should not follow some easily detected, and exploited, pattern. A player can avoid predictability by literally randomizing each period. An alternative, pursued in Hu (2014), is the idea that even if a pure strategy exhibits a pattern that can be detected and exploited in principle, it may be impossible to do so in practice if the pattern is sufficiently complicated.

A subtlety with the subjective interpretation is that if there are three or more players, then two players might have different subjective beliefs about what a third player might do. This is assumed away in our notation, where $\sigma_i$ does not depend on anything having to do with the other players.

### 3.5 The Reduced Strategic Form.

The strategic form may contain pure strategies that are “redundant.”

Consider Figure 6, which gives the game box for a variant of Battle of the Sexes (Figure 3) that has an added third strategy, $B'$, for player 1. $B'$ is redundant in the sense that it generates the same payoffs as $B$ for either player, regardless of what

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<td>$B'$</td>
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Figure 6: An alternate game box for Battle of the Sexes.
player 2 does. From the standpoint of payoff maximization, neither player cares whether player 1 chooses \( B' \) rather than \( B \), or vice versa.

Now consider Figure 7, another Battle of the Sexes variant. Pure strategy \( C \) is also redundant, in the sense that player 1 can generate the same payoffs, for either player, as \( C \) by randomizing 50:50 between \( A \) and \( B \). \( C \) is, in effect, simply the name of a particular mixed strategy.

Given a game in strategic form, the reduced strategic form is the game in which strategies that are redundant in the above senses have been removed. Arguably, the redundant strategies just add notational clutter, and can be excluded without affecting the analysis of the game in any meaningful way. There are, however, two caveats.

First, the claim isn’t that redundant strategies would never be played but rather that the reduced form can be analyzed first and then any redundant strategies can be brought back into consideration, if for some reason this is desired. For example, in Battle of the Sexes, if the analysis of the reduced form (Figure 6) predicts that player 1 would play strategy \( B \), then the same analysis predicts that in the variant game of Figure 6, player 1 might play either \( B \) or \( B' \), or might even randomize between them.

Second, when we get to extensive from games, there will be solution concepts, such as subgame perfect equilibrium, that, strictly speaking, require using the full strategy set generated by the extensive form, even if some of those strategies turn out to be redundant in the above sense.

4 Nash Equilibrium.

4.1 The Best Response Correspondence.

Given a profile of opposing (mixed) strategies \( \sigma_{-i} \in \Sigma_{-i} \), let \( \text{BR}_i(\sigma_{-i}) \) be the set of mixed strategies for player \( i \) that maximize player \( i \)'s expected payoff; formally,

\[
\text{BR}_i(\sigma_{-i}) = \{ \sigma_i \in \Sigma_i : \forall \hat{\sigma}_i \in \Sigma_i, u_i(\sigma_i, \sigma_{-i}) \geq u_i(\hat{\sigma}_i, \sigma_{-i}) \}.
\]

An element of \( \text{BR}_i(\sigma_{-i}) \) is called a best response to \( \sigma_{-i} \).

Given a profile of strategies \( \sigma \in \Sigma \), let \( \text{BR}(\sigma) \) be the set of mixed strategy profiles \( \hat{\sigma} \) such that, for each \( i \), \( \hat{\sigma}_i \) is a best response to \( \sigma_{-i} \). Formally,

\[
\text{BR}(\sigma) = \{ \hat{\sigma} \in \Sigma : \forall i \hat{\sigma}_i \in \text{BR}_i(\sigma_{-i}) \}.
\]

\[
\begin{array}{c|cc}
 & A & B \\
\hline
A & 8,10 & 0,0 \\
B & 0,0 & 10,8 \\
C & 4,5 & 5,4 \\
\end{array}
\]

Figure 7: An alternate game box for Battle of the Sexes.
BR is a correspondence on $\Sigma$. Since $\Sigma_i$ is compact for each $i$, $\Sigma$ is compact. For each $i$, expected payoffs are continuous, which implies that for any $\sigma \in \Sigma$ and any $i$, $BR_i(\sigma_{-i})$ is not empty. Thus, BR is a non-empty-valued correspondence on $\Sigma$.

The definition of best response builds in an assumption that players are expected payoff maximizers; see also the discussion in Section 3.3. One consequence of expected payoff maximization is that payoffs can be rescaled in certain ways without affecting the best response correspondence.

1. For any $s_{-i}$, one can add any constant to $u_i(s_i, s_{-i})$, the same constant for every $s_i$ but possibly differing across $s_{-i}$, without affecting player $i$’s best response correspondence.

2. One can multiply $u_i$ by any positive constant, the same constant for every $s_i$ and every $s_{-i}$, without affecting player $i$’s best response correspondence.

Example 7. The game in which player 1’s payoffs are

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
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</thead>
<tbody>
<tr>
<td>T</td>
<td>5</td>
<td>11</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>14</td>
</tr>
</tbody>
</table>

generates the same best response correspondence for player 1 as the game with payoffs

<table>
<thead>
<tr>
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<th>L</th>
<th>R</th>
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</thead>
<tbody>
<tr>
<td>T</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

To see this, subtract 2 from the first column, 11 from the second, and then multiply all payoffs by 1/3. □

On the other hand, a non-linear transformation of payoffs will typically change the best response correspondence. In particular, if the transformation is concave, then the best response correspondence will embody risk aversion, as in the next example.

Example 8. Consider the game in which player 1’s payoffs are

<table>
<thead>
<tr>
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<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>M</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

If player 2 randomizes equally over L or R then the set of player 1’s best responses is any mixture over T and M, for an expected payoff of 5.

If we take the square root, a concave transformation, then payoffs are (approximately),

<table>
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<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>3.2</td>
<td>0</td>
</tr>
<tr>
<td>M</td>
<td>0</td>
<td>3.2</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>
In this case, if player 2 randomizes equally over $L$ or $R$, then player 1’s unique best response is $B$.

For more on this issue, see Weinstein (2016). □

4.2 Nash Equilibrium.

The single most important solution concept for games in strategic form is Nash equilibrium (NE). Nash himself called it an “equilibrium point.” I discuss some aspects of the intellectual history of Nash equilibrium in Section 5.

A NE is a (mixed) strategy profile $\sigma^*$ such that for each $i$, $\sigma^*_i$ is a best response to $\sigma^*_{-i}$, hence $\sigma^* \in \text{BR}(\sigma^*)$: a NE is a fixed point of BR.

**Definition 1.** Fix a game. A strategy profile $\sigma^* \in \Sigma$ is a Nash equilibrium (NE) iff $\sigma^*$ is a fixed point of BR. If $\sigma^*$ is a NE then the associated vector of expected payoffs is a Nash equilibrium payoff profile.

I will go through a number of examples of NE shortly.

As a point of terminology, it is important to remember that the Nash equilibrium is a strategy profile, not a payoff profile.

A NE is *pure* if all the strategies are pure. It is *fully mixed* if every $\sigma_i$ is fully mixed. If a NE is neither pure nor fully mixed, then it is *partly mixed*. Strictly speaking, a pure NE is a special case of a mixed NE; in practice, however, I may sometimes (sloppily) write mixed NE when what I really mean is fully or partly mixed.

In separate notes (Game Theory Basics III), I survey some motivating stories for NE. It is easy to be led astray by motivating stories, however, and confuse what NE “ought” to be with what NE, as a formalism, actually is. For the moment, therefore, I focus narrowly on the formalism.

**Remark 3.** From Section 4.1, we know that certain types of payoff transformations do not affect the best response correspondence. It follows that these transformations also do not affect the set of Nash equilibria. □

The following result was first established in Nash (1950a).

**Theorem 1.** Every (finite) strategic form game has at least one NE.

**Proof.** For each $i$, $\Sigma_i$ is compact and convex and hence $\Sigma$ is compact and convex. BR is a non-empty valued correspondence on $\Sigma$ and it is easy to show that it has a closed, convex graph. By the Kakutani fixed point theorem, BR has a fixed point $\sigma \in \text{BR}(\sigma)$. ■

Theorem 1 is not true if we restrict attention to pure strategies. Rock-Paper-Scissors (Section 2) is a canonical example of a game that has no pure strategy NE. Instead, RPS has a unique NE that is fully mixed and in it both players randomize equally across their three strategies; see also Example 11.
Remark 4. Non-finite games may not have NE. A trivial example is the game in which you name a number \( \alpha \in [0,1) \), and I pay you a billion dollars with probability \( \alpha \) (and nothing with probability \( 1-\alpha \)). Assuming that you prefer more (expected) money to less, you don’t have a best response and hence there is no NE.

The difficulty in the above example was that the strategy set was not compact. A related example has \( \alpha \in [0,1] \), which is now compact, but with payoff of zero if you choose \( \alpha = 1 \). Here the problem is that payoff function is not continuous.

The most straightforward extension of Theorem 1 is to games in which each player’s strategy set is a compact subset of some measurable topological space and the payoff functions are continuous. In this case, the set of mixed strategies is weak-* compact. One can take finite approximations to the strategy sets, apply Theorem 1 to get a NE in each finite approximation, appeal to compactness to get a convergent subsequence of these mixed strategy profiles, and then argue, via continuity of payoffs, that the limit must be an equilibrium in the original game.

This approach will not work if the utility function is not continuous and, unfortunately, games with discontinuous utility functions are fairly common in economics. A well known example is the Bertrand duopoly game, discussed in Example 16 in Section 4.4. For a survey of the literature on existence of NE in general and also on existence of NE with important properties (e.g., pure strategy NE or NE with monotonicity properties), see Reny (2008).

Remark 5. A related point is the following. The Brouwer fixed point theorem states that if \( D \subseteq \mathbb{R}^N \) is compact and convex and \( f: D \to D \) is continuous then \( f \) has a fixed point: there is an \( x \in D \) such that \( f(x) = x \). Brouwer is the underlying basis for the Kakutani fixed point theorem, which was used in the proof Theorem 1, and one can also use Brouwer more directly to prove NE existence (as, indeed, Nash himself subsequently did for finite games in Nash (1951)). Thus, Brouwer implies NE.

The converse is also true: given a compact convex set \( D \subseteq \mathbb{R}^N \) and a continuous function \( f: D \times D \), one can construct a game in which the strategy set for player 1 is \( D \) and, if a NE exists, it must be that \( s_1 = f(s_1) \). One such construction can be found here.

4.3 NE and randomization.

The following fact says that a best response gives positive probability to a pure strategy only if that pure strategy is, in its own right, also a best response.

**Theorem 2.** For any finite game and for any \( \sigma_{-i} \), if \( \sigma_i \in BR_i(\sigma_{-i}) \) and \( \sigma_i(s_i) > 0 \) then \( s_i \in BR_i(\sigma_{-i}) \).

**Proof.** Since the set of pure strategies is assumed finite, one of these strategies, call it \( s^*_i \), has highest expected payoff (against \( \sigma_{-i} \)) among all pure strategies. Let the expected payoff of \( s^*_i \) be \( c^*_i \). For any mixed strategy \( \sigma_i \), the expected payoff is
the convex sum of the expected payoffs to $i$’s pure strategies. Therefore, $c_i^*$ is the highest possible payoff to any mixed strategy for $i$, which implies that, in particular, $s^*_i$ is a best response, as is any other pure strategy that has an expected payoff of $c_i^*$.

I now argue by contraposition. Suppose $s_i \notin BR_i(\sigma_{-i})$, hence $u_i(s_i, \sigma_{-i}) < c_i^*$. If $\sigma_i[s_i] > 0$, then $u_i(\sigma_i, \sigma_{-i}) < c_i^*$, hence $\sigma_i \notin BR_i(\sigma_{-i})$, as was to be shown. ■

Theorem 2 implies that a NE mixed $\sigma_i$ will give positive probability to two different pure strategies only if those pure strategies each earn the same expected payoff as the mixed strategy. That is, in a NE, a player is *indifferent* between all of the pure strategies that he plays with positive probability. This provides a way to compute mixed strategy NE, at least in principle.

*Example 9* (Finding mixed NE in $2 \times 2$ games). Consider a general $2 \times 2$ game given by the game box in Figure 8. Let $q = \sigma_2(L)$. Then player 1 is indifferent between $T$ and $B$ iff

$$qa_1 + (1-q)b_1 = qc_1 + (1-q)d_1,$$

or

$$q = \frac{d_1 - b_1}{(a_1 - c_1) + (d_1 - b_1)},$$

provided, of course, that $(a_1 - c_1) + (d_1 - b_1) \neq 0$ and that this fraction is in $[0, 1]$.

Similarly, if $p = \sigma_1(T)$, then player 2 is indifferent between $L$ and $R$ if

$$pa_2 + (1-p)c_2 = pb_2 + (1-p)d_2,$$

or

$$p = \frac{d_2 - c_2}{(a_2 - b_2) + (d_2 - c_2)},$$

again, provided that $(a_1 - c_1) + (d_1 - b_1) \neq 0$ and that this fraction is in $[0, 1]$.

Note that the probabilities for player 1 are found by making the other player, player 2, indifferent. A common error is to do the calculations correctly, but then to write the NE incorrectly, by flipping the player roles. □

*Example 10* (NE in Battle of the Sexes). Battle of the Sexes (Example 3) has two pure strategy NE: $(A, B)$ and $(B, A)$. There is also a mixed strategy NE with

$$\sigma_1(A) = \frac{8 - 0}{(10 - 0) + (8 - 0)} = \frac{4}{9}$$
and
\[
\sigma_2(A) = \frac{10 - 0}{(10 - 0) + (8 - 0)} = \frac{5}{9}
\]
which I can write as \(((\frac{4}{9}, \frac{5}{9}), (\frac{5}{9}, \frac{4}{9}))\); where \((\frac{4}{9}, \frac{5}{9})\), for example, means the mixed strategy in which player 1 plays \(A\) with probability \(4/9\) and \(B\) with probability \(5/9\).

\[\square\]

**Example 11 (NE in Rock-Paper-Scissors).** In Rock-Scissor-Paper (Section 2), the NE is unique and in it each player randomizes 1/3 each across all three strategies. That is, the unique mixed strategy NE profile is

\[
\left( \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right), \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \right).
\]

It is easy to verify that if either player randomizes 1/3 across all strategies, then the other payer is indifferent across all three strategies. \[\square\]

**Example 12.** Although every pure strategy that gets positive probability in a NE is a best response, the converse is not true: in a NE, there may be best responses that are not given positive probability. A trivial example where this occurs is a game where the payoffs are constant, independent of the strategy profile. Then players are always indifferent and every strategy profile is a NE.

### 4.4 More NE examples.

**Example 13.** Figure 9 provides the game box for a game called Matching Pennies, which is simpler than Rock-Paper-Scissors but in the same spirit. This game has

\[
\begin{array}{cc}
H & T \\
H & 1, -1 & -1, 1 \\
T & -1, 1 & 1, -1 \\
\end{array}
\]

Figure 9: Matching Pennies.

one equilibrium and in this equilibrium both players randomize 50:50 between the two pure strategies. \[\square\]

**Example 14.** Figure 10 provides the game box for one of the best known games in game theory, the Prisoner’s Dilemma (PD) (the PD is actually a set of similar games; this is one example). PD is a stylized depiction of friction between joint incentives (the sum of payoffs is maximized by \((C, C)\)) and individual incentives (each player has incentive to play \(D\)).

It is easy to verify that \(D\) is a best response no matter what the opponent does. The unique NE is thus \((D, D)\).

A PD-like game that shows up fairly frequently in popular culture goes something like this. A police officer tells a suspect, “We have your partner, and he’s
already confessed. It will go easier for you if you confess as well.” Although there
is usually not enough detail to be sure, the implication is that it is (individually)
payoff maximizing not to confess if the partner also does not confess; it is only if
the partner confesses that it is payoff maximizing to confess. If that is the case, the
game is not a PD but something like the game in Figure 11, where I have written
payoffs in a way that makes them easier to interpret as prison sentences (which are
bad).

\[
\begin{array}{cc}
C & D \\
C & 4, 4 & 0, 6 \\
D & 6, 0 & 1, 1 \\
\end{array}
\]

Figure 10: A Prisoner’s Dilemma.

This game has three NE: (Silent, Silent), (Confess, Confess), and a mixed NE in
which the suspect stays silent with probability 1/3. In contrast, in the (true) PD,
each player has incentive to play D regardless of what he thinks the other player
will do. □

Example 15 (Cournot Duopoly). The earliest application of game theoretic reason-
ing that I know of to an economic problem is the quantity-setting duopoly model of
Augustin Cournot, in Cournot (1838).

There are two players (firms). For each firm \( S_i = \mathbb{R}_+ \cup \{\text{Out}\} \). Note that this
is not a finite game; one can work with a finite version of the game but analysis
of the continuum game is easier in many respects. For the pure strategy profile
\((q_1, q_2) \in \mathbb{R}_+^2 \) (\( q_i \) for “quantity”), the payoff to player \( i \) is,

\[(1 - q_1 - q_2)q_i - k,\]

where \( k > 0 \) is a small number (in particular, \( k < 1/16 \)), provided \( q_1 + q_2 \leq 1. \)
If \( q_1 + q_2 > 1 \) then the payoff to firm \( i \) is \(-k\). If a firm chooses Out instead of a
quantity then it has a payoff of 0. Note, in particular, that in this formulation a
firm gets a payoff of \(-k\) if it chooses \( q_i = 0 \) rather than Out. If one firm chooses
Out while the other chooses a quantity, then the latter firm operates as a monopoly.

The idea here is that both firms produce a quantity \( q_i \) of a good, best thought
of as perishable, and then the market clears somehow, so that each receives a price
\( 1 - q_1 - q_2 \), provided the firms don’t flood the market (\( q_1 + q_2 > 1 \)); if they flood the
market, then the price is zero. I have assumed that the firm’s payoff is its profit. There is a fixed cost \((k, \text{ for ease of notation taken to be the same for both firms})\) but no marginal costs.

Ignore \(k\) for a moment. It is easy to verify that for any probability distribution over \(q_2\), provided \(E[q_2] < 1\), the best response for firm 1 is the pure strategy, \(\frac{1 - E[q_2]}{2}\).

If \(E[q_2] \geq 1\), then the market is flooded, in which case any \(q_1\) generates the same profit, namely 0. Because of this, if \(k = 0\), then there are an infinite number of “nuisance” NE in which both firms flood the market by producing at least 1 in expectation. This is the technical reason for including \(k > 0\) (which seems realistic in any event): for \(k > 0\), if \(E[q_2] \geq 1\), then the best response is to shut down (choose Out rather than a quantity). Adding a small marginal cost of production would have a similar effect. As an aside, this illustrates an issue that sometimes arises in economic modeling: sometimes one can simplify the analysis, at least in some respects, by making the model more complicated.\(^2\)

With \(k \in (0, 1/16)\), the NE is unique and in it both firms produce \(1/3\).\(^3\) One can find this by noting that, in pure strategy equilibrium, \(q_1 = (1 - q_2)/2\) and \(q_2 = (1 - q_1)/2\); this system of equations has a unique symmetric solution with \(q_1 = q_2 = q = (1 - q)/2\), which implies \(q = 1/3\).

Total output in NE is thus \(2/3\) (and price is \(1/3\)). The NE of the Cournot quantity game is intermediate between monopoly (where quantity is \(1/2\)) and efficiency (in the sense of maximizing total surplus; since there is zero marginal cost, efficiency calls for total output of 1). □

**Example 16 (Bertrand Duopoly).** Writing in the 1880s, about 50 years after Cournot, Joseph Bertrand objected to Cournot’s quantity-setting model of duopoly on the grounds that quantity competition is unrealistic. In Bertrand’s model, firms set prices rather than quantities.

Assume (to make things simple) that there are no production costs, either marginal or fixed. I allow \(p_i\), the price charged by firm \(i\), to be any number in \(\mathbb{R}^+\), so, once again, this is *not* a finite game.

- \(p_i \geq 1\) then firm \(i\) gets a payoff of 0, regardless of what the other firm does. Assume henceforth that \(p_i \leq 1\) for both firms.
- If \(p_1 \in (0, p_2)\) then firm 1 receives a payoff of \((1 - p_1)p_1\). In words, firm 1 gets the entire demand at \(p_1\), which is \(1 - p_1\). Revenue, and hence profit, is \((1 - p_1)p_1\). Implicit here is the idea that the good of the two firms are

---

\(^2\)If \(E[q_2]\) is not defined, then player 1’s best response does not exist; by definition, this cannot happen in NE, so I ignore this case.

\(^3\)For \(k \in [1/16, 1/9]\), there are asymmetric NE in which one firm produces the monopoly level, namely \(1/2\), and the other firm chooses Out.
homogeneous and hence perfect substitutes: consumers buy which ever good is cheaper; they are not interested in the name of the seller.

- If \( p_1 = p_2 = p \), both firms receive a payoff of \((1 - p)p/2\). In words, the firms split the market at price \( p \).

- If \( p_1 > p_2 \), then firm 1 gets a payoff of 0.

Payoffs for firm 2 are defined symmetrically. It is not hard to see that firm 1’s best response is as follows.

- If \( p_2 > 1/2 \) then firm 1’s best response is \( p_1 = 1/2 \).

- If \( p_2 \in (0, 1/2] \) then firm 1’s \textit{best response set is empty}. Intuitively, firm 1 wants to undercut \( p_2 \) by some small amount, say \( \varepsilon > 0 \), but \( \varepsilon > 0 \) can be made arbitrarily small. Technically, the existence failure for best response arises because of a discontinuity in the payoff function. One can address the existence failure by requiring that prices lie on a finite grid (be expressed in pennies, for example), but this introduces other issues.

- If \( p_2 = 0 \), then any \( p_1 \in \mathbb{R}_+ \) is a best response.

Notwithstanding the fact that best response isn’t defined over much of the domain, there is a unique pure NE (and in fact the unique NE period), namely the price profile \((0,0)\).

The Bertrand model thus predicts the competitive result with only two firms. At first this may seem paradoxical, but remember that I am assuming that the goods are homogeneous. The bleak prediction of the Bertrand game (bleak from the viewpoint of the firms; consumers do well and the outcome is efficient) suggests that firms have strong incentive to differentiate their products, and also strong incentive to collude (which may be possible if the game is repeated, as I discuss later in the course).

\textit{Example 17}. Francis Edgeworth, writing in the 1890s, about ten years after Bertrand, argued that firms often face capacity constraints, and these constraints can affect equilibrium analysis. Continuing with the Bertrand duopoly of Example 16, suppose that firms can produce up to \( 1/2 \) at zero cost but then face infinite cost to produce any additional output. This would model a situation in which, for example, each firm had stockpiled a perishable good; this period, a firm can sell up to the amount stockpiled, but no more, at zero opportunity cost (zero since any amount not sold melts away in the night).

Suppose that sales of firm 1 (taking into consideration the capacity constraint
and ignoring the trivial case with $p_1 > 1$ are given by,

$$q_1(p_1, p_2) = \begin{cases} 
0 & \text{if } p_1 > p_2 \text{ and } p_1 > 1/2, \\
1/2 - p_1 & \text{if } p_1 > p_2 \text{ and } p_1 \leq 1/2, \\
(1 - p_1)/2 & \text{if } p_1 = p_2 \leq 1, \\
1 - p_1 & \text{if } p_1 < p_2 \text{ and } p_1 > 1/2, \\
1/2 & \text{if } p_1 < p_2 \text{ and } p_1 \leq 1/2.
\end{cases}$$

The sales function for firm 2 is analogous.

For interpretation, suppose that there are a continuum of consumers, with a consumer whose “name” is $\ell \in [0, 1]$ wanting to buy at most one unit of the good at a price of at most $v_\ell = 1 - \ell$. Then, integrating across consumers, market demand facing a monopolist would be $1 - p$, as specified.

With this interpretation, consider the case $p_1 > p_2$ and $p_1 \leq 1/2$. Then $p_2 < 1/2$ and demand for firm 2 is $1 - p_2 > 1/2$. But because of firm 2’s capacity constraint, it can only sell $1/2$, leaving $(1 - p_2) - 1/2 > 0$ potentially for firm 1, provided firm 1’s price is not too high. Suppose that the customers who succeed at buying from firm 2 are the customers with the highest valuations (they are the customers willing to wait on line, for example). Then the leftover demand facing firm 1, called residual demand, is $1/2 - p_1$.

Note that this specification of residual demand is a substantive assumption about how demand to the low priced firm gets rationed. We could instead have assumed that customers are assigned to firm 2 randomly. That would have resulted in a different model.

In contrast to the Bertrand model, it is no longer an equilibrium for both firms to charge $p_1 = p_2 = 0$. In particular, if $p_2 = 0$, then firm 1 is better off setting $p_1 = 1/4$, which is the monopoly price for residual demand. In fact, there is no pure strategy equilibrium here. There is, however, a symmetric mixed strategy equilibrium, which can be found as follows.

Note first that the maximum profit from residual demand is $1/16 (= 1/4 \times 1/4)$. A firm can also get a profit of $1/16$ by being the low priced firm at a price of $1/8$ (and selling $1/2$). The best response for firm 1 to pure strategies of firm 2 is then,

$$\begin{align*}
1/2 & \text{ if } p_2 > 1/2, \\
\emptyset & \text{ if } p_2 \in (1/8, 1/2], \\
1/4 & \text{ if } p_2 \leq 1/8.
\end{align*}$$

The best response for firm 2 to pure strategies of player 1 is analogous.

Intuitively, the mixed strategy equilibrium will involve a continuous distribution, and not put positive probability on any particular price, because of the price undercutting argument that is central to the Bertrand game. So, I look for a symmetric equilibrium in which firms randomize over an interval of prices. The obvious interval to look at is $[1/8, 1/4]$. 

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For it to be an equilibrium for firm 1 to randomize over the interval \([1/8, 1/4]\), firm 1 has to get the same profit from any price in this interval.\(^4\) The profit at either \(p_1 = 1/8\) (where the firm gets capacity constrained sales of 1/2) or \(p_1 = 1/4\) (where the firm sells 1/2-1/4=1/4 to residual demand) is 1/16. Thus for any other \(p_1 \in (1/8, 1/4)\) it must be that, letting \(F_2(p_1)\) be the probability that player 2 charges a price less than or equal to \(p_1\),

\[
F_2(p_1)p_1(1/2 - p_1) + (1 - F_2(p_1))p_1(1/2) = 1/16,
\]

where the first term is profit when \(p_2 < p_1\) and the second term is profit when \(p_2 > p_1\); because the distribution is continuous, the probability that \(p_2 = p_1\) is zero and so I can ignore this case.

Solving for \(F_2\) yields

\[
F_2(p_1) = \frac{1}{2p_1} - \frac{1}{16p_1^2}.
\]

The symmetric equilibrium then has \(F_1 = F_2\), where \(F_2\) is given by the above expression. The associated density is

\[
f_2(p_1) = \frac{1}{8p_1^3} - \frac{1}{2p_1^2}
\]

and is graphed in Figure 17. This makes some intuitive sense. If we imagine this equilibrium being repeated, then most of the time prices will be close to 1/8, reflecting the undercutting logic of the Bertrand game, but from time to time one of the prices will be close to 1/4.

\(^4\)In a continuum game like this, this requirement can be relaxed somewhat, for measure theoretic reasons, but this is a technicality as far as this example is concerned.
Edgeworth argued that in a situation such as this, with no pure strategy equilibrium, prices would just bounce around. Nash equilibrium makes a more specific prediction: the distribution of prices is generated by the mixed strategy equilibrium.

There is an intuition, first explored formally in Kreps and Scheinkman (1983), that the qualitative predictions of Cournot’s quantity model can be recovered in a more realistic, multi-period setting in which firms make output choices in period 1 and then sell that output in period 2. The Edgeworth example that I just worked through is an example of what the period 2 analysis can look like. If the capacity constraints had been, instead, 1/3 each (the equilibrium quantities in the Cournot game) then you can verify that the equilibrium of the Bertrand-Edgeworth game would, in fact, have been for both firms to charge a price of 1/3. □

4.5 The MinMax Theorem.

Recall from Section 2 that a two-player game is zero-sum iff $u_2 = -u_1$. A game is constant-sum iff there is a $c \in \mathbb{R}$ such that $u_1 + u_2 = c$.

Theorem 4 below, called the MinMax Theorem (or, more frequently, the Min-Max Theorem), was one of the first important general results in game theory. It was established for zero-sum games in von Neumann (1928). Although, it is rare to come across zero-sum, or constant-sum, games in economic applications, the MinMax Theorem is useful because it is sometimes possible to provide a quick proof of an otherwise difficult result by constructing an auxiliary constant-sum game and then applying the MinMax Theorem to this new game; an example of this is given by the proof of Theorem 6 in Section 4.8.

The MinMax Theorem is an almost immediate corollary of the following result.

**Theorem 3.** In a finite constant-sum game, if $\sigma^* = (\sigma_1^*, \sigma_2^*)$ is a NE then,

$$u_1(\sigma^*) = \max_{\sigma_1} \min_{\sigma_2} u_1(\sigma) = \min_{\sigma_2} \max_{\sigma_1} u_1(\sigma).$$

**Proof.** Suppose that $u_1 + u_2 = c$. Let $V = u_1(\sigma^*)$. Since $\sigma^*$ is a NE, and since, for player 2, maximizing $u_2 = c - u_1$ is equivalent to minimizing $u_1$,

$$V = \max_{\sigma_1} u_1(\sigma_1, \sigma_2^*) = \min_{\sigma_2} u_1(\sigma_1^*, \sigma_2).$$

(1)

Then,

$$\min_{\sigma_2} \max_{\sigma_1} u_1(\sigma) \leq \max_{\sigma_1} u_1(\sigma_1, \sigma_2^*) = \min_{\sigma_2} u_1(\sigma_1^*, \sigma_2) \leq \max_{\sigma_1} \min_{\sigma_2} u_1(\sigma).$$

(2)

On the other hand, for any $\tilde{\sigma}_2$,

$$u_1(\sigma_1^*, \tilde{\sigma}_2) \leq \max_{\sigma_1} u_1(\sigma_1, \tilde{\sigma}_2),$$

Yes, von Neumann (1928) is in German; no, I haven’t read it in the original.
hence,
\[
\min_{\sigma_2} u_1(\sigma_1^*, \sigma_2) \leq \max_{\sigma_1} u_1(\sigma_1, \bar{\sigma}_2).
\]
Since the latter holds for every \(\bar{\sigma}_2\),
\[
\min_{\sigma_2} u_1(\sigma_1^*, \sigma_2) \leq \min_{\sigma_1} \max_{\sigma_2} u_1(\sigma_1, \sigma_2).
\]
Combining with expressions (1) and (2),
\[
V = \min_{\sigma_2} u_1(\sigma_1^*, \sigma_2) = \min_{\sigma_1} \max_{\sigma_2} u_1(\sigma).
\]
By an analogous argument,
\[
V = \max_{\sigma_1} u_1(\sigma_1, \sigma_2^*) = \max_{\sigma_1} \min_{\sigma_2} u_1(\sigma).
\]
And the result follows. ■

Theorem 4 (MinMax Theorem). In any finite constant-sum game, there is a number \(V\) such that
\[
V = \max_{\sigma_1} \min_{\sigma_2} u_1(\sigma_1, \sigma_2^*) = \min_{\sigma_2} \max_{\sigma_1} u_1(\sigma).
\]
Proof. By Theorem 1, there is NE, \(\sigma^*\). Set \(V = u_1(\sigma^*)\) and apply Theorem 3. ■

Remark 6. The MinMax Theorem can be proved using a separation argument, rather than a fixed point argument (the proof above implicitly uses a fixed point argument, since it invokes Theorem 1, which uses a fixed point argument). □

By way of interpretation, one can think of \(\max_{\sigma_1} \min_{\sigma_2} u_1(\sigma)\) as the payoff that player 1 can guarantee for herself, no matter what player 2 does. Theorem 3 implies that, in a constant-sum game, player 1 can guarantee an expected payoff equal to her NE payoff. This close connection between NE and what players can guarantee for themselves does not generalize to non-constant-sum games, as the following example illustrates.

Example 18. Consider the game in Figure 13, which originally appeared in Aumann and Maschler (1972). This game is not constant sum. In this game, there is a unique NE, with \(\sigma_1^* = (3/4, 1/4)\) and \(\sigma_2^* = (1/2, 1/2)\) and the equilibrium payoffs are \(u_1(\sigma^*) = 1/2\) and \(u_2(\sigma^*) = 3/4\).

\[
\begin{array}{cc}
L & R \\
T & 1,0 & 0,1 \\
B & 0,3 & 1,0 \\
\end{array}
\]

Figure 13: The Aumann-Maschler Game.
But in this game, $\sigma^*_1$ does not guarantee player 1 a payoff of 1/2. In fact, if player 1 plays $\sigma^*_1$ then player 2 can hold player 1’s payoff to 1/4 by playing $\sigma_2 = (0, 1)$ (i.e., playing $R$ for certain); it is not payoff maximizing for player 2 to do this, but it is feasible. Player 1 can guarantee a payoff of 1/2, but doing so requires randomizing $\sigma_1 = (1/2, 1/2)$, which is not a strategy that appears in any Nash equilibrium of this game. □

4.6 Never a Best Response and Rationalizability.

Definition 2. Fix a game. A mixed strategy $\sigma_i$ is never a best response (NBR) iff there does not exist a profile of opposing mixed strategies $\sigma_{-i}$ such that $\sigma_i \in \text{BR}_i(\sigma_{-i})$.

If a pure strategy $s_i$ is NBR then it cannot be part of any NE. Therefore, it can be, in effect, deleted from the game when searching for NE. This motivates the following procedure.

Let $S^1_i \subseteq S_i$ denote the strategy set for player $i$ after deleting all pure strategies that are never a best response. $S^1_i$ is non-empty since, as noted in Section 4.1, the best response correspondence is not empty-valued. The $S^1_i$ form a new game, and for that game we can ask whether any strategies are NBR. Deleting those strategies, we get $S^2_i \subseteq S^1_i \subseteq S_i$, which again is not empty. And so on.

Since the game is finite, this process terminates after a finite number of rounds in the sense that there is a $t$ such that for all $i$, $S^t_{i+1} = S^t_i$. Let $S^R_i$ denote this terminal $S^t_i$ and let $S^R = \text{product of the } S^R_i$. The strategies $S^R_i$ are called the rationalizable strategies for player $i$; $S^R$ is the set of rationalizable (pure) strategy profiles. Informally, these are all the strategies that make sense after introspective reasoning of the form, “$s_i$ would maximize my expected payoffs if my opponents were to play $\sigma_{-i}$, and $\sigma_{-i}$ would maximize my opponents’ payoffs (for each opponent, independently) if they thought . . . .” Rationalizability was first introduced into game theory in Bernheim (1984) and Pearce (1984).

Theorem 5. Fix a game. If $\sigma$ is a NE and $\sigma_i[s_i] > 0$ then $s_i \in S^R_i$.

Proof. Almost immediate by induction and Theorem 2. ■

Remark 7. The construction of $S^R$ uses maximal deletion of never a best response strategies at each round of deletion. In fact, this does not matter: it is not hard to show that if, whenever possible, one deletes at least one strategy for at least one player at each round, then eventually the deletion process terminates, and the terminal $S^t$ is $S^R$. □

Example 19. In the Prisoner’s Dilemma (Example 14, Figure 10), $C$ is NBR. $D$ is the unique rationalizable strategy. □
Example 20 (Rationalizability in the Cournot Duopoly). Recall the Cournot Duopoly (Example 15).

Suppose $k \in (0, 1/16)$. Then no pure strategy greater than 1/2 is a best response. Therefore, in the first round of deletion, delete all strategies in $(1/2, \infty)$; we can actually delete more, as discussed below, but we can certainly delete this much. In the second round of deletion, one can delete all strategies in $[0, 1/4)$.

This generates a nested sequence of compact intervals. There is no termination in finite time, as for a finite game, but one can show that the intersection of all these intervals is the point $\{1/3\}$. Moreover, at some point in this process of iterated deletion, Out becomes strictly dominated. Therefore, $S^R_i = \{1/3\}$ for both firms. And, indeed, the NE is, as we already saw, $(1/3, 1/3)$.

In fact, because of $k > 0$, firm 1 strictly prefers Out if $E[q_2]$ is too large; more precisely, at the first round of deletion, firm 1 prefers Out if $q_2 > 1 - 2\sqrt{k}$. So we can also eliminate all $q_1$ in $[0, \sqrt{k})$, even at the first round. But note that this complication does not undermine the argument above; it only states that we could have eliminated more strategies at each round. □

A natural, but wrong, intuition is that any strategy in $S^R_i$ gets positive probability in at least one NE.

Example 21. Consider the game in Figure 14. $S^R_i = \{A, B, C\}$ for either player.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>4,4</td>
<td>1,1</td>
<td>1,1</td>
</tr>
<tr>
<td>B</td>
<td>1,1</td>
<td>-2,2</td>
<td>-2,2</td>
</tr>
<tr>
<td>C</td>
<td>1,1</td>
<td>-2,2</td>
<td>2, -2</td>
</tr>
</tbody>
</table>

Figure 14: Rationalizability and NE.

But the unique NE is $(A, A)$. □

4.7 Correlated Rationalizability.

Somewhat abusing notation, let $BR_i(\sigma^c_{-i})$ denote the set of strategies for player $i$ that are best responses to the distribution $\sigma^c_{-i} \in \Sigma^c_{-i} = \Delta(S_{-i})$.

Definition 3. Fix a game. A mixed strategy $\sigma_i$ is correlated never a best response (CNBR) iff there does not exist a distribution over opposing strategies $\sigma^c_{-i}$ such that $\sigma_i \in BR_i(\sigma^c_{-i})$.

Again, by Theorem 2, if a pure strategy $s_i$ is correlated never a best response, then so is any mixed strategy that gives $s_i$ positive probability.

\footnotemark[6]
The set of CNBR strategies is a subset of the set of NBR strategies. In particular, any CNBR strategy is NBR (since a strategy that is not a best response to any distribution is, in particular, not a best response to any independent distribution). If $N = 2$ the CNBR and NBR sets are equal (trivially, since in this case there is only one opponent). If $N \geq 3$, the CNBR set can be a proper subset of the NBR set. In particular, one can construct examples in which a strategy is NBR but not CNBR (see, for instance Fudenberg and Tirole (1991)). The basic intuition is that it is easier for a strategy to be a best response to something the larger the set of allowed distributions over opposing strategies, and the set of correlated distributions is larger than the set of independent distributions.

If one iteratively deletes CNBR strategies, one gets, for each player $i$, the set $S^C_{iR}$ of correlated rationalizable strategies. Since, at each step of the iterative deletion, the set of CNBR is a subset of the set of NBR strategies, $S^R_i \subseteq S^C_{iR}$ for all $i$, with equality if $N = 2$. Let $S^C_{iR}$ denote the set of correlated rationalizable profiles. The set of distributions over strategies generated by Nash equilibria is a subset of the set of distributions over $S^R$, which in turn is a subset of the set of distributions over $S^C_{iR}$.

4.8 Strict Dominance.

**Definition 4.** Fix a game.

- Mixed strategy $\sigma_i$ strictly dominates mixed strategy $\hat{\sigma}_i$ iff for any profile of opposing pure strategies $s_{-i}$,

$$u_i(\sigma_i, s_{-i}) > u_i(\hat{\sigma}_i, s_{-i}).$$

- Mixed strategy $\hat{\sigma}_i$ is strictly dominated iff there exists a mixed strategy that strictly dominates it.

If a mixed strategy $\sigma_i$ strictly dominates every other strategy, pure or mixed, of player $i$ then it is called strictly dominant. It is easy to show that a strictly dominant strategy must, in fact, be pure.

**Definition 5.** Fix a game. Pure strategy $s_i$ is strictly dominant iff it strictly dominates every other strategy.

**Example 22.** Recall the Prisoner’s Dilemma of Example 14 and Figure 10. Then, for either player, $C$ is strictly dominated by $D$; $D$ is strictly dominant. $\Box$

**Theorem 6.** Fix a game. For each player $i$, a strategy is strictly dominated iff it is CNBR.

**Proof.** $\Rightarrow$. If $\hat{\sigma}_i$ strictly dominates $\sigma_i$ then $\hat{\sigma}_i$ has strictly higher payoff than $\sigma_i$ against any distribution over opposing strategies (since expected payoffs are a convex combination of payoffs against pure strategy profiles).
\(\Rightarrow\). Suppose that \(\hat{\sigma}_i\) is CNBR. Construct a zero-sum game in which the two players are \(i\) and \(-i\), the strategy set for \(i\) is \(S_i\) (unless \(\hat{\sigma}_i\) happens to be pure, in which case omit that), and the strategy set for \(-i\) is \(S_{-i}\). In this new game, a mixed strategy for player \(-i\) is a probability distribution over \(S_{-i}\), which (if there are two or more opponents) is a correlated strategy \(\sigma^c_{-i}\) in the original game. In the new game, the payoff to player \(i\) is,

\[\tilde{u}_i(s) = u_i(s_i, s_{-i}) - u_i(\hat{\sigma}_i, s_{-i}).\]

and the payoff to \(-i\) is \(\tilde{u}_{-i} = -\tilde{u}_i\).

Let

\[V = \min_{\sigma^c_{-i}} \max_{\sigma_i} \tilde{u}_i(\sigma_i, \sigma^c_{-i}).\]

Since \(\hat{\sigma}_i\) is CNBR, it follows that for every \(\sigma^c_{-i}\) there is a \(\sigma_i\) for which \(\tilde{u}_i(\sigma_i, \sigma^c_{-i}) > 0\). Therefore \(V > 0\). By Theorem 4,

\[V = \max_{\sigma_i} \min_{\sigma^c_{-i}} \tilde{u}_i(\sigma_i, \sigma^c_{-i}).\]

This implies that there exists a \(\sigma^*_i\) (in fact, a NE strategy for player \(i\) in the new game) such that for all \(s_{-i} \in S_{-i}\), \(\tilde{u}_i(\sigma^*_i, s_{-i}) > 0\), hence

\[u_i(\sigma^*_i, s_{-i}) > u_i(\hat{\sigma}_i, s_{-i}),\]

so that \(\sigma^*_i\) strictly dominates \(\hat{\sigma}_i\). \(\blacksquare\)

In the following examples, \(N = 2\), so the CNBR and NBR strategy sets are equal.

**Example 23.** Consider the game in Figure 15 (I’ve written payoffs only for player 1); I also used this game in Example 8 in Section 4.1. \(T\) is the best response if the probability of \(L\) is more than 1/2. \(M\) is the best response if the probability of \(L\) is less than 1/2. If the probability of \(L\) is exactly 1/2 the the set of best responses is the set of mixtures over \(T\) and \(M\). Hence \(B\) is NBR.

\(B\) is strictly dominated for player 1 by the mixed strategy \((1/2, 1/2, 0)\) (which gets a payoff of 5 no matter what player 2 does, whereas \(B\) always gets 4). There is no strictly dominant strategy.

Note that \(B\) is not strictly dominated by any pure strategy. So it is important in this example that I allow for dominance by a mixed strategy. \(\blacksquare\)
Example 24. Consider the game in Figure 16. $T$ is the best response if the probability of $L$ is more than $3/5$ and $M$ is the best response if the probability of $L$ is less than $2/5$. If the probability of $L$ is in $(2/5, 3/5)$ then $B$ is the best response. If the probability of $L$ is $3/5$ then the set of best responses is the set of mixtures over $T$ and $B$. If the probability of $L$ is $2/5$ then the set of best responses is the set of mixtures over $M$ and $B$.

Note that $B$ is not a best response to any pure strategy for player 2. So it is important in this example that I allow for mixed strategies by player 2.

$B$ is not strictly dominated for player 1, hence (trivially) there is no strictly dominant strategy. □

It follows from Theorem 6 that one can find the correlated rationalizable strategy sets, $S^{CR}$, by iteratively deleting strictly dominated strategies. Once again, this may help in narrowing down the search for Nash equilibria.

4.9 How many NE are there?

Fix players and strategy sets and consider all possible assignments of payoffs. With $N$ players and $|S|$ pure strategy profiles, payoffs can be represented as a point in $\mathbb{R}^{N|S|}$. Wilson (1971) proved that there is a precise sense in which the set of Nash equilibria is finite and odd (which implies existence, since 0 is not an odd number) for most payoffs in $\mathbb{R}^{N|S|}$. As we have already seen, Rock-Paper-Scissors has one equilibrium, while Battle of the Sexes has three. A practical implication of the finite and odd result, and the main reason why I am emphasizing this topic in the first place, is that if you are trying to find all the equilibria of the game, and so far you have found four, then you are probably missing at least one.

Although the set of NE in finite games is typically finite and odd, there are exceptions.

Example 25 (A game with two NE). A somewhat pathological example is the game in Figure 17. This game has exactly two Nash equilibria, $(A,A)$ and $(B,B)$. In particular, there are no mixed strategy NE: if either player puts any weight on $A$ than the other player wants to play $A$ for sure. □

The previous example seems somewhat artificial, and in fact it is extremely unusual in applications to encounter a game where the set of NE is finite and even. But it is quite common to encounter strategic form games that have an infinite number
of NE; such games are common when the strategic form represents an extensive form game with a non-trivial temporal structure.

Example 26 (An Entry Deterrence Game). Consider the game in Figure 18. For motivation, player 1 is considering entering a new market (staying in or going out). Player 2 is an incumbent who, if there is entry, can either accommodate (reach some sort of status quo intermediate between competition and monopoly) or fight (start a price war).

There are two pure strategy NE, \((I,A)\) and \((O,F)\). In addition, there are a continuum of NE of the form, player 1 plays \(O\) and player 2 accommodates, should entry nevertheless occur, with probability \(q \leq 1/16\). More formally, every profile of the form \((O, (q, 1-q))\) is a NE for any \(q \in [0, 1/16]\). Informally, the statement is that it is a NE for the entrant to stay out provided that the incumbent threatens to fight with high enough probability (accommodate with low enough probability).

It is standard in game theory to argue that the only reasonable NE of this game is the pure strategy NE \((I, A)\). One justification for this claim is that if player 1 puts any weight at all on \(I\) then player 2’s best response is to play \(A\) for sure, in which case player 1’s best response is likewise to play \(I\) for sure. The formal version of this statement is to say that only \((I, A)\) is an admissible NE, because the other NE put positive weight on a weakly dominated strategy, namely \(F\); I discuss weak dominance later in the course. For the moment, what I want to stress is that every one of this infinity of NE is a NE, whether we find it reasonable or not. □

Assuming that the number of equilibria is in fact finite and odd, how many equilibria are typical? In \(2 \times 2\) games, the answer is 1 or 3. What about in general? A lower bound on how large the set of NE can possibly be in finite games is provided by \(L \times L\) games (two players, each with \(L\) strategies) in which, in the game box representation, payoffs are \((1, 1)\) along the diagonal and \((0, 0)\) elsewhere. In such games, there are \(L\) pure strategy Nash equilibria, corresponding to play along the diagonal, and an additional \(2^L - (L+1)\) fully or partly mixed NE. The total number
of NE is thus $2^L - 1$. This example is robust; payoffs can be perturbed slightly and there will still be $2^L - 1$ NE.

This is extremely bad news. First, it means that the maximum number of NE is growing exponentially in the size of the game. This establishes that the general problem of calculating all of the equilibria is computationally intractable. Second, it suggests that the problem of finding even one NE may, in general, be computationally intractable; for work on this, see Daskalakis, Goldberg and Papadimitriou (2008). Note that the issue is not whether algorithms exist for finding one equilibrium, or even all equilibria. For finite games, there exist many such algorithms. The problem is that the time taken by these algorithms to reach a solution can grow explosively in the size of the game. For results on the number of equilibria in finite games, see McLennan (2005).

4.10 The structure of NE.

As in Section 4.9, if players and strategy sets are fixed, then payoff functions can be represented as a vector in $\mathbb{R}^{N|S|}$, giving payoffs for each player and each strategy profile. Let

$$\mathcal{N} : \mathbb{R}^{N|S|} \to \mathcal{P}(\Sigma)$$

be the NE correspondence: for any specification of payoffs $u \in \mathbb{R}^{N|S|}$, $\mathcal{N}(u)$ is the set of NE for the game defined by $u$. By Theorem 1, $\mathcal{N}$ is non-empty-valued.

The following result states that the limit of a sequence of NE is a NE (in particular, the set of NE for a fixed $u$ is closed) but that for some games there are NE for which there are no nearby NE in some nearby games.

**Theorem 7.** $\mathcal{N}$ is upper hemicontinuous but (for $|S| \geq 2$) not lower hemicontinuous.

**Proof.**

1. **Upper Hemicontinuity.** Since $\Sigma$ is compact, it suffices to prove that $\mathcal{N}$ has a closed graph, for which it suffices to prove that every point $(u, \sigma)$ in the complement of graph($\mathcal{N}$) is interior. Take any $(u, \sigma)$ in the complement of graph($\mathcal{N}$). Then there is an $i$ and a pure strategy $s_i$ for which

$$u_i(s_i, \sigma_{-i}) - u_i(\sigma_i, \sigma_{-i}) > 0.$$  

By continuity of expected utility in both $u_i$ and $\sigma$, this inequality holds for all points within a sufficiently small open ball around $(u, \sigma)$, which completes the argument.

2. **Lower Hemicontinuity.** Consider the game in figure 19. For any $\varepsilon > 0$, the unique NE is $(A, L)$. But if $\varepsilon = 0$ then any mixed strategy profile is a NE, and in particular $(A, R)$ is a NE. Therefore, for any sequence of $\varepsilon > 0$ converging to zero, there is no sequence of NE converging to $(A, R)$. This example implies a failure of lower hemicontinuity in any game with $|S| \geq 2$. 

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Figure 19: The NE correspondence is not lower hemicontinuous.

The set of NE need not be connected, let alone convex. But one can prove that for any finite game, the set of NE can be decomposed into a finite number of connected components; see Kohlberg and Mertens (1986). In the Battle of the Sexes (Example 10 in Section 4.3), there are three components, namely the three NE. In the entry deterrence game of Example 26 (Section 4.9), there are two components: the pure strategy \((\text{In},A)\) and the component of \((O, (q, 1−q))\) for \(q \in [0, 1/16]\).

5 Some Game Theory History.

The concept of Nash equilibrium seems quite natural, and indeed it had been anticipated by Cournot over a hundred years earlier. This raises the question of why Nash equilibrium is named in honor of Nash rather than, say, Cournot. As already noted, Nash himself just referred to it as an “equilibrium point.” The following comments are based on Nachbar and Weinstein (2016), written for a special section of the Notices of the American Mathematical Society honoring Nash after his death; see.

In proposing his solution concept, Nash made two methodological contributions. First, there was confusion prior to Nash about what was a model and what was a solution to that model. It was common, for example, to see references to Cournot equilibrium and Bertrand equilibrium, as though Cournot and Bertrand were using different equilibrium concepts, and there was even a vexed literature called conjectural variations that tried to formalize this. Nash’s position, which is now generally accepted as correct, was that one should try always to apply the same solution concept. In particular, Cournot equilibrium and Betrand equilibrium refer to the same solution concept, what we now call Nash equilibrium, applied to different models; see Mayberry, Nash and Shubik (1953).

Second, prior to Nash, the state of the art in dealing with games was the von-Neumann-Morgenstern (VN-M) solution, developed in von Neumann and Morgenstern (1947). From a modern (post-Nash) perspective, the VN-M solution is a mash-up of two different approaches to game theory, one for strategic form games and another for games in coalition form. As I also mentioned in the Introduction, the coalition form abstracts away from the details about what individual players do and focuses instead on what payoff allocations are physically possible, both for all players taken together and for subsets of players. Because of its hybrid nature, the

\[
\begin{array}{cc}
L & R \\
A & 1, \varepsilon & 1, 0 \\
\end{array}
\]
VN-M solution sometimes obscures strategic issues. For example, in the Prisoner’s Dilemma, the payoff profiles predicted by the VN-M solution are all Pareto efficient; in particular, the VN-M solution rules out the payoff profile corresponding to the Nash equilibrium (in strictly dominant strategies, no less), namely \((D, D)\). In effect, the VN-M solution assumes away the tension between joint payoff maximization and individual opportunism that the Prisoner’s Dilemma is designed to illustrate. More broadly, it is nearly impossible to use the VN-M solution to study incentives, one of the central topics of modern economics research. Nash argued that game theory should maintain a division between the study of strategic form and coalition form games, and Nash offered what we now call Nash equilibrium as a solution concept for games in strategic form. The Nash program, see Serrano (2008), bridges the two approaches by investigating whether, or under what circumstances, the analysis of strategic form games gives the same qualitative answers as the analysis of coalition form games.

Remark 8. Nash called the analysis of games in strategic form “non-cooperative game theory” and the analysis of games in coalition form “cooperative game theory.” This terminology has stuck, which is unfortunate, I think, because it sets up a false expectation that the strategic form rules out cooperation, which is not correct, as the Nash program, alluded to above, underscores. □

References


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