1 Metric Spaces Basics.

1.1 Metric spaces.

A metric space \((X, d)\) consists of a set of points, \(X\), together with a function \(d : X \times X \to \mathbb{R}^+_0\), called a metric. The interpretation is that \(d(x, y)\) is the distance between \(x\) and \(y\). To qualify as a metric, the function \(d\) must satisfy certain properties.

**Definition 1.** Given a set \(X\), a function \(d : X \times X \to \mathbb{R}^+_0\) is a metric iff, for any \(x, y, z \in X\), the following properties hold.

1. \(d(x, y) = 0\) iff \(x = y\).
2. \(d(x, y) = d(y, x)\).
3. \(d(x, z) \leq d(x, y) + d(y, z)\).

The third property is called the **triangle inequality**.

The triangle inequality gets its name because, in standard geometry, if the three points \(x\), \(y\), and \(z\) form a triangle then the length of the side from \(x\) to \(z\) is less then the sum of the lengths of the other two sides.

Think of \(X\) as fundamental while the metric \(d\) is an overlay, like the latitude and longitude grid on a map. Any \(X\) has an infinity of possible metrics. At a minimum, metrics can differ because of units (inches rather than centimeters). But it is also possible for different metrics to give different answers to questions like, “which is greater, the distance from \(x\) to \(y\) or the distance from \(x\) to \(z\)?” Which metric I choose depends entirely on which is most helpful to me. For some spaces, notably \(X = \mathbb{R}^N\), there is a default metric that is convenient for almost any application. For other spaces, such as variants of \(X = \mathbb{R}^\omega\), there is no default metric.

Let \((X, d)\) be a metric space and let \(A \subseteq X\). Unless I state explicitly otherwise, I assume that \(A\) inherits the \(d\) metric, with domain restricted to \(A \times A\). \(A\) is then a **metric subspace** of \(X\).
1.2 Metrics derived from norms.

Many, although not all, of the metric spaces used in economics are derived from normed vector spaces. Any norm \( f \) induces a metric via \( d_f(x, y) = f(x - y) \).

**Theorem 1.** Let \( X \) be a normed vector space, with norm \( f \). Define \( d_f : X \times X \to \mathbb{R}_+ \) by \( d_f(x, y) = f(x - y) \). Then \( d_f \) is a metric.

**Proof.** I must check that the three metric properties hold.

1. \( d_f(x, y) = 0 \) iff \( x = y \), since \( f(x - y) = 0 \) iff \( x - y = 0 \) iff \( x = y \).
2. \( d_f(x, y) = d_f(y, x) \) since \( f(x - y) = | -1| f(x - y) = f(y - x) \).
3. \( d_f(x, z) \leq d_f(x, y) + d_f(y, z) \) since \( f(x - z) = f(x - y + y - z) \leq f(x - y) + f(y - z) \).

\[ \blacksquare \]

In particular, if \( d_f \) is derived from a norm \( f \) then \( d_f(x, 0) = f(x) \). In general, when dealing with a normed vector space, it is implicit that the metric used is the one derived from the norm.

1.3 Examples of Metric Spaces.

1.3.1 \( \mathbb{R}^N \) with the Euclidean metric.

Recall that if \( x \in \mathbb{R}^N \) then the Euclidean norm of \( x \) is

\[ \|x\| = (x \cdot x)^{1/2}. \]

If \( N = 1 \) then \( \|x\| = |x| \). The notes on \( \mathbb{R}^N \) established that \( \|x\| \) does indeed satisfy the required norm properties.

The Euclidean metric is then defined by,

\[ d(x, y) = \|x - y\| = [(x - y) \cdot (x - y)]^{1/2} = \sqrt{\sum_n (x_n - y_n)^2}. \]

By Theorem 1, \( d \) is a metric.

Whenever I write \( \mathbb{R}^N \), you can assume that the metric is the Euclidean metric unless I explicitly state otherwise. The Euclidean metric is also the default when I work with subsets of \( \mathbb{R}^N \), such as the interval \((0, 1]\) or \( \mathbb{Q} \) (the set of rational numbers).
1.3.2 $\mathbb{R}^N$ with the max metric.

Although $d$ is the default metric for $\mathbb{R}^N$, there are many (infinitely many) other possible metrics. Recall the max norm from the notes on normed vector spaces.

$$\|x\|_{\text{max}} = \max_n |x_n|.$$ 

The notes on normed vector spaces verify that the max norm is indeed a norm.

Define the max metric as,

$$d_{\text{max}}(x, y) = \|x - y\|_{\text{max}} = \max_n |x_n - y_n|.$$ 

Since, the max norm is a norm, Theorem 1 implies that $d_{\text{max}}$ is, indeed, a metric.

1.3.3 $\mathbb{R}^N$ with the discrete metric.

For $x, y \in \mathbb{R}^N$, define

$$d_d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

This is called the discrete metric. It is easy to verify that it is a true metric. It is not, however, generated by any norm and has a number of peculiar properties. It is useful mainly for counterexamples.

1.3.4 $\ell^\infty$ with the sup metric.

Recall, from the notes on normed vector spaces, that $\mathbb{R}^\omega$ is the set of points of the form $(x_1, x_2, x_3, \ldots)$, where each $x_n \in \mathbb{R}$. Also recall that $\ell^\infty$ is the set of points in $\mathbb{R}^\omega$ that are bounded in the sense that $x \in \ell^\infty$ iff there is an $M$ (which can depend on $x$) such that for all $n$, $|x_n| < M$. For $x \in \ell^\infty$, the sup norm, given by,

$$\|x\|_{\text{sup}} = \sup_n |x_n|$$

is well defined. The notes on normed vector spaces establish that $\ell^\infty$ is a vector space and that $\| \cdot \|_{\text{sup}}$ is indeed a norm.

Define the sup metric as,

$$d_{\text{sup}}(x, y) = \|x - y\|_{\text{sup}} = \sup_n |x_n - y_n|.$$ 

By Theorem 1, $d_{\text{sup}}$ is, indeed, a metric.
1.3.5 $\mathbb{R}^\omega$ with the $d_{pw}$ metric.

For any $x, y \in \mathbb{R}^\omega$, define

$$d_{pw}(x, y) = \sup_n \min\{1, |x_n - y_n|\}.$$  

(This metric has no standard name. I am calling it $d_{pw}$ where the “pw” is mnemonic for “pointwise”; the reason for this name is addressed in the notes on Completeness and Compactness in $\mathbb{R}^\omega$.)

Loosely, there are two differences between $d_{pw}$ and $d_{sup}$. First, $d_{pw}$ weights high-numbered coordinates less than low numbered coordinates. Second, $d_{pw}$ truncates the coordinate by coordinate measure of distance to a maximum value of 1. Because of the truncation, $d_{pw}$, unlike $d_{sup}$, is defined on all of $\mathbb{R}^\omega$.

**Theorem 2.** $d_{pw}$ is a metric on $\mathbb{R}^\omega$.

**Proof.** Exercise. ■

$d_{pw}$ is not generated by a norm. In particular, the norm would have to be $f(x) = d_{pw}(x, 0)$. But this $f$ violates norm property 2. For example, if $x = (1, 1, \ldots)$ and $y = (2, 2, \ldots)$, then $f(x) = f(y) = 1$.

2 Balls.

Let $(X, d)$ be a metric space. For $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, the open $\varepsilon$ ball around $x$ is

$$N_\varepsilon(x) = \{a \in X : d(x, a) < \varepsilon\}.$$  

Think of $N_\varepsilon$ as mnemonic for “nearby within $\varepsilon$.”

**Example 1.** For the Euclidean metric on $\mathbb{R}^N$. In $\mathbb{R}$, $N_\varepsilon(x)$ is simply the interval $(x - \varepsilon, x + \varepsilon)$. In $\mathbb{R}^2$, for the Euclidean metric, $N_\varepsilon(x)$ is a disk of radius $\varepsilon$, centered at $x$, excluding the boundary circle. In $\mathbb{R}^3$, $N_\varepsilon(x)$ is a solid ball of radius $\varepsilon$, centered at $x$, excluding the boundary sphere. □

**Example 2.** For the Euclidean metric on $[0, 1] \subseteq \mathbb{R}$, if $\varepsilon < 1$, $N_\varepsilon(0) = [0, \varepsilon)$, which I will also write as $(-\varepsilon, \varepsilon) \cap \mathbb{R}$, with $(-\varepsilon, \varepsilon)$ understood to denote the full interval in $\mathbb{R}$ of length $2\varepsilon$. □

**Example 3.** For the Euclidean metric on $\mathbb{Q}$, $N_\varepsilon(0) = (-\varepsilon, \varepsilon) \cap \mathbb{Q}$, again with $(-\varepsilon, \varepsilon)$ understood to denote the full interval in $\mathbb{R}$ of length $2\varepsilon$. □

**Example 4.** For the max metric on $\mathbb{R}^2$, $N_\varepsilon(x)$ is the interior of a square with sides of length $2\varepsilon$, centered at $x$. In $\mathbb{R}^3$, $N_\varepsilon(x)$ is the interior of a cube with sides of length $2\varepsilon$, centered at $x$. □

**Example 5.** For the discrete metric on $\mathbb{R}$, for any $x \in \mathbb{R}$, if $\varepsilon \leq 1$ then $N_\varepsilon(x) = \{x\}$ while if $\varepsilon > 1$ then $N_\varepsilon(x) = \mathbb{R}$. □
Remark 1. In a vector space $X$, a set $A \subseteq X$ is convex iff for any two points $x, y \in A$ and any $\theta \in [0, 1]$, $\theta x + (1 - \theta)y \in A$; in words, a set is convex iff for any two points in $A$, the line segment joining them is also in $A$. Recall (from the notes on normed vector spaces) that a function $f : X \to \mathbb{R}$ is convex iff for any $x, y \in X$, and any $\theta \in [0, 1]$, $f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$. A simple example of a convex function is absolute value on $\mathbb{R}$: $f(x) = |x|$.

It is easy to prove that if a function is convex then for any $b \in \mathbb{R}$, the set \{x $\in X : f(x) < b$\} (which may be empty) is convex. In the case of a metric based on a norm, this implies that, since any norm is a convex function (see the notes on normed vector spaces), any $\varepsilon$ ball is convex. □

3 Bounded Sets and Totally Bounded Sets.

3.1 Basic Definitions.

Let $(X, d)$ be a metric space. $A \subseteq X$ is bounded iff there is an $x \in X$ and $r > 0$, such that $A \subseteq N_r(x)$. In words, $A$ is bounded iff it is contained in some ball of large enough radius. In a metric vector space, one can always take $x$ to be the origin.$^2$

A set $A$ is totally bounded iff for any $\varepsilon > 0$ there is a finite set of points $E \subseteq X$ such that

$$A \subseteq \bigcup_{x \in E} N_\varepsilon(x).$$

If $A \subseteq \bigcup_{x \in E} N_\varepsilon(x)$ then the set of balls $\{N_\varepsilon(x)\}_{x \in E}$ is said to cover $A$. In words, then, $A$ is totally bounded iff, for any $\varepsilon > 0$ no matter how small, $A$ can be covered by a finite number of open balls of radius $\varepsilon$ (the number of balls required will typically go up as $\varepsilon$ shrinks).

I leave it as an exercise to show that $A$ is totally bounded iff for any $\varepsilon > 0$ there is a finite set $E \subseteq A$ such that $\{N_\varepsilon(x)\}_{x \in E}$ covers $A$ (the difference being that $E \subseteq A$ rather than $E \subseteq X$).

The next result establishes that any totally bounded set is bounded.

Theorem 3. Let $(X, d)$ be a metric space. If $A \subseteq X$ is totally bounded then it is bounded.

Proof. Since $A$ is totally bounded there is a finite set $E \subseteq X$ such that $A$ is covered by the balls $N_1(x)$ for $x \in E$. Fix any $x^* \in X$ and let $r = \max_{x \in E} d(x^*, x) + 1$. I claim that for any $a \in A$, $a \in N_r(x^*)$, which shows that $A$ is bounded. For any $a \in A$, there is an $x_a \in E$ such that $a \in N_1(x_a)$. By the triangle inequality, $d(x^*, a) \leq d(x^*, x_a) + d(x_a, a) < d(x^*, x_a) + 1 \leq r$. ■

$^2$Suppose that $A \subseteq N_r(x)$. Let $\hat{r} = r + d(x, 0)$. Then, by the triangle inequality, for any $a \in A$, $d(a, 0) \leq d(a, x) + d(x, 0) < r + d(x, 0) = \hat{r}$. Therefore, $A \subseteq N_{\hat{r}}(0)$.

5
The converse of Theorem 3 is true in $\mathbb{R}^N$, as recorded in Section 3.2 (the proof is deferred to Section 6), but not in general, as illustrated in Section 3.3.

\textbf{Remark 2.} It is not hard to show that a set $A$ is totally bounded iff for any $\varepsilon > 0$, $A$ can be covered by a finite set of $\varepsilon$ balls that have centers in $A$ (rather than, as in the definition, centers in $X$). $\square$

3.2 Bounded and Totally Bounded Sets in $\mathbb{R}^N$.

\textbf{Theorem 4.} If $A \subseteq \mathbb{R}^N$ is bounded then it is totally bounded.

\textbf{Proof.} Deferred to Section 6. $\blacksquare$

As illustrated in Section 3.3, in metric spaces other than $\mathbb{R}^N$, bounded sets may not be totally bounded. The proof of Theorem 4 in Section 6 hints at what can go wrong: for a fixed $r$, the number of open $\varepsilon$ balls used to cover an open $r$ ball in $\mathbb{R}^N$ is on the order of $(r/\varepsilon)^N$, and in particular, if $\varepsilon < r$, goes to infinity as $N$ goes to the infinity.

3.3 Bounded and Totally Bounded Sets in $(\ell^\infty, d_{sup})$.

\textbf{Theorem 5.} In $(\ell^\infty, d_{sup})$ there exist bounded sets that are not totally bounded.

\textbf{Proof.} It suffices to provide an example. Let $A = \{(1,0,0,\ldots), (0,1,0,\ldots), \ldots \}$. $A$ is bounded; in particular, $A \subseteq N_2(0)$. But it is not totally bounded. The $d_{sup}$ distance between any two points in $A$ is 1, so at most one point of $A$ can be in any $d_{sup}$ ball of radius 1/2 (and hence diameter 1). Since there are infinitely many points in $A$, there is no finite set $E$ such that $A \subseteq \bigcup_{x \in E} N_{1/2}(x)$. $\blacksquare$

4 Sequences and Completeness.

4.1 Sequences.

Let $(X,d)$ be a metric space. An infinite sequence is a function from the natural numbers to $X$. The value that this function takes at 0 is denoted $x_0$, the value at 1 is denoted $x_1$, and so on. The domain is called the set of indices. One can use other sets of integers (including negative numbers) as indices. In particular, I typically use the natural numbers starting at 1, rather than 0. And one can work with finite sequences. All sequences here will be infinite, however. I denote a typical sequence by $(x_t) = (x_1, x_2, \ldots, x_t, \ldots)$.$^3$

$^3$Some texts, including earlier versions of these notes, use $\{x_t\} = \{x_1, x_2, \ldots \}$ as notation for a sequence. This notation creates confusion with sets. For example, the sequence $(1,1,\ldots)$ and the set $\{1,1,\ldots\} = \{1\}$ are very different objects.
A sequence in $X$ can be interpreted as a point in the countably infinite Cartesian product of $X$: both are defined as a function from the natural numbers to $X$. For this reason, I am using the same notation for both the sequence $(x_1, x_2, \ldots)$ in $X$ and the point $(x_1, x_2, \ldots)$ in $X \times X \times \ldots$. In particular, a point in $\mathbb{R}^\omega$ can be interpreted as a sequence of real numbers. $\mathbb{R}^\omega$ is, therefore, sometimes called a sequence space. With this interpretation, a sequence of points in $\mathbb{R}^\omega$ is a sequence of sequences. Since this quickly becomes confusing, I typically refer to an element of $\mathbb{R}^\omega$ as a point rather than as a sequence.

I use subscripts to denote both elements of a sequence and, in the case of sequences in $\mathbb{R}^N$ or $\mathbb{R}^\omega$, coordinates of points in the sequence. Thus, if $(x_t)$ is a sequence in $\mathbb{R}^N$ then $x_{tn}$ is the $n$ coordinate of the $t$ term of the sequence. For example, if the sequence is $((0, 1), (1, 5), (-1, 2), \ldots)$ then $x_{3,1}$ is the first coordinate of the third term in the sequence, namely $-1$. This notation is awkward, but the alternatives are equally awkward.

The range of a sequence is the set of values that the sequence takes; formally, the range is the range of the function that defines the sequence. A sequence is said to be in a set $C \subseteq X$ iff the range of the sequence is a subset of $C$. A sequence is bounded iff its range is bounded.

### 4.2 Convergence.

A sequence $(x_t)$ in $X$ is said to converge to a point $x \in X$, also written

$$x_t \rightarrow x$$

or $\lim_{t} x_t = x$, iff for every $\epsilon > 0$ there is a $T \in \mathbb{N}$ such that for all $t > T$, $x_t \in B_\epsilon(x)$ (equivalently, $d(x_t, x) < \epsilon$). In words, $x_t \rightarrow x$ iff all terms late in the sequence are close to $x$.

Although establishing facts about convergence for important classes of sequences is a major theme in basic analysis texts, I will not do so here. It is useful, however, to note that the sequence $(1/t)$ in $\mathbb{R}$ converges to 0. To see this, note that, by the Archimedean Property of $\mathbb{R}$, for any $\epsilon > 0$ there is a $T \in \mathbb{N}$ such that $T > 1/\epsilon$, hence $1/T < \epsilon$. This implies that $1/t < \epsilon$ for all $t > T$, hence $1/t \rightarrow 0$.

A related observation is that it suffices to check convergence only for a strictly positive sequence $(\epsilon_s)$ that converges to 0; any such sequence will do. In view of the discussion in the previous paragraph, a prime candidate is $\epsilon_s = 1/s$.

#### Theorem 6.

Let $(X, d)$ be a metric space, let $(x_t)$ be a sequence in $X$ and let $x \in X$. Consider any strictly positive sequence $(\epsilon_s)$ in $\mathbb{R}$ such that $\epsilon_s \rightarrow 0$. Then $x_t \rightarrow x$ iff for any $s$ there is a $T$ such that for all $t > T$, $x_t \in B_{\epsilon_s}(x)$.

**Proof.** $\Rightarrow$. This is immediate since any $\epsilon_s$ is simply one possible $\epsilon$.

$\Leftarrow$. Take any $\epsilon > 0$. Since $\epsilon_s \rightarrow 0$, there is an $s$ such that $\epsilon_s < \epsilon$. Take any $T$ such that for all $t > T$, $x_t \in B_{\epsilon_s}(x)$. For any such $t$, since $\epsilon_s < \epsilon$, $x_t \in B_{\epsilon}(x)$. ■
In a metric space, a sequence cannot converge to two different points.

**Theorem 7.** Let \((x_t)\) be a sequence in a metric space \((X, d)\). If \(x_t \to x^*\) and \(x_t \to \hat{x}\) then \(x^* = \hat{x}\).

**Proof.** Fix any \(\varepsilon > 0\). If \(x_t \to x^*\) then there is a \(T^*\) such that for all \(t > T^*\), \(d(x_t, x^*) < \varepsilon/2\). If \(x_t \to \hat{x}\) then there is a \(\hat{T}\) such that for all \(t > \hat{T}\), \(d(x_t, \hat{x}) < \varepsilon/2\).

Therefore, by the triangle inequality, for any \(t > \max\{T^*, \hat{T}\}\),
\[
d(x^*, \hat{x}) \leq d(x^*, x_t) + d(x_t, \hat{x}) < \varepsilon.
\]

Since \(\varepsilon\) was arbitrary, this implies \(d(x^*, \hat{x}) = 0\), hence \(x^* = \hat{x}\). ■

**Remark 3.** For sequences in \(\mathbb{R}\), it is common to write \(x_t \to \infty\), and to say that \(x_t\) **diverges to infinity**, iff for any \(M \in \mathbb{R}\) there is a \(T\) such that for all \(t > T\), \(x_t > M\). Similarly, it is common to write \(x_t \to -\infty\), and to say that \(x_t\) **diverges to negative infinity**, iff for any \(M \in \mathbb{R}\) there is a \(T\) such that for all \(t > T\), \(x_t < M\).

Note that \(x_t \to \infty\) does **not** mean that for any \(\varepsilon > 0\) there is a \(T\) such that for all \(t > T\), \(x_t \in N_\varepsilon(\infty)\), since \(N_\varepsilon(\infty)\) has no meaning. □

**Remark 4.** The space \(X\) matters for the definition of convergence. Consider, for example, the sequence of rational numbers \((x_t) = (1, 0, 1, 0, \ldots)\) that is generated by the decimal expansion of \(\sqrt{2}\). It is not hard to show that \(x_t \to \sqrt{2}\) if \(X = \mathbb{R}\). But if \(X = \mathbb{Q}\) then \((x_t)\) does **not** converge. □

### 4.3 Subsequences.

Given a sequence \((x_t)\), one can construct a new sequence, called a subsequence, out of the original sequence. The subsequence is of the form \((x_{t_k}) = (x_{t_1}, x_{t_2}, \ldots)\) where \(t_1 < t_2 < \ldots\). Even if \((x_t)\) fails to converge, some of its subsequences may converge.

**Example 6.** Consider the sequence \((x_t) = (1, 0, 1, 0, \ldots)\) in \(\mathbb{R}\). This sequence does not converge. I can form the subsequence \((x_{t_k}) = (x_1, x_3, x_5, \ldots) = (1, 1, 1, \ldots)\). This subsequence converges to 1. There are, of course, other subsequences. For example, consider \((x_{10}, x_{11}, x_{20}, x_{21}, \ldots) = (0, 1, 0, 1, \ldots)\). This does not converge. □

**Theorem 8.** Let \((X, d)\) be a metric space. A sequence \((x_t)\) in \(X\) converges to a point \(x^* \in X\) iff every subsequence of \((x_t)\) converges to \(x^*\).

**Proof.** Exercise. ■
4.4 Cauchy sequences and completeness.

**Definition 2.** Let \((X,d)\) be a metric space. A sequence \((x_t)\) in \(X\) is Cauchy iff for any \(\varepsilon > 0\) there is a \(T\) such that for any \(s, t > T\), \(d(x_s, x_t) < \varepsilon\).

In words, a sequence is Cauchy iff all terms late in the sequence are close to each other. The next theorem establishes that all convergent sequences are Cauchy.

**Theorem 9.** Let \((X,d)\) be a metric space. Let \((x_t)\) be a sequence in \(X\). If \((x_t)\) is convergent then it is Cauchy.

**Proof.** Suppose \(x_t \to x\). Consider any \(\varepsilon > 0\). Choose \(T\) such that for all \(t > T\), \(d(x_t, x) < \varepsilon/2\). Then, by the triangle inequality, for any \(s, t > T\), \(d(x_s, x_t) \leq d(x_s, x) + d(x, x_t) < \varepsilon\). ■

If the converse is always true, if Cauchy sequences always converge, then the set is called complete.

**Definition 3.** Let \((X,d)\) be a metric space. \(X\) is complete iff for any sequence \((x_t)\) in \(X\), if \((x_t)\) is Cauchy then there is a point \(x \in X\) such that \(x_t \to x\).

If \((X,d)\) is a metric space and \(A \subseteq X\) then the statement that \(A\) is complete means that \(A\) is complete when viewed as a metric subspace of \(X\). I will refer to \(A\) as either a complete space or a complete set.

Later in the course, I establish that \(\mathbb{R}\) is complete. In contrast, the space of rational numbers, \(\mathbb{Q}\), is not complete.

**Example 7.** Consider the Cauchy sequence \((3, 3.1, 3.14, 3.141, \ldots)\). Considered as a sequence in \(\mathbb{R}\), this sequence converges to \(\pi\). But \(\pi \not\in \mathbb{Q}\). As a sequence in the metric space \(\mathbb{Q}\), therefore, this Cauchy sequence does not converge. □

To sum up, informally, a Cauchy sequence is a sequence that looks like it converges. If the sequence lies in a complete space then a Cauchy sequence actually does converge. But a Cauchy sequence in a space such as \(\mathbb{Q}\) that is not complete need not converge.

For later use, I record the following additional facts about Cauchy sequences.

**Theorem 10.** Let \((X,d)\) be a metric space. Let \((x_t)\) be a Cauchy sequence in \(X\). Then \((x_t)\) is bounded.

**Proof.** Since \((x_t)\) is Cauchy, there is a \(T\) such that for all \(t, s > T\), \(d(x_s, x_t) < 1\). In particular, for all \(t > T\), \(d(x_t, x_{T+1}) < 1\). Let

\[
r = \max\{d(x_1, x_{T+1}), \ldots, d(x_T, x_{T+1})\} + 1
\]

Then the range of \((x_t)\) lies in \(N_r(x_{T+1})\). ■

Note that, combined with Theorem 9, this implies that any convergent sequence is bounded.
**Theorem 11.** Let \((X, d)\) be a metric space. Let \((x_t)\) be a Cauchy sequence in \(X\). If \((x_t)\) has a convergent subsequence then it is convergent.

**Proof.** Suppose \(x_t \rightarrow x\) and choose any \(\varepsilon > 0\). Since \(x_t \rightarrow x\), there is a \(K\) such that if \(k > K\) then \(x_t \in N_{\varepsilon/2}(x)\). Since \((x_t)\) is Cauchy there is a \(T\) such that if \(s, t > T\), \(d(x_s, x_t) < \varepsilon/2\). Then, by the triangle inequality, for any \(t > T\) and any \(t_k > \max\{T, t_K\}\)

\[
d(x_t, x) \leq d(x_t, x_{t_k}) + d(x_{t_k}, x) < \varepsilon.
\]

\[\blacksquare\]

**Remark 5.** Completeness as used here refers to Cauchy completeness. There are other forms of completeness used in mathematics, notably Dedekind completeness. See Propp (2014). \(\square\)

5 **Open and Closed Sets.**

5.1 **Open sets.**

**Definition 4.** Let \((X, d)\) be a metric space and let \(A \subseteq X\). A point \(x \in A\) is interior to \(A\) iff it is contained in a ball inside \(A\): there is an \(\varepsilon > 0\) such that \(N_{\varepsilon}(x) \subseteq A\).

**Theorem 12.** Let \((X, d)\) be a metric space and consider any \(O \subseteq X\). The following are equivalent.

1. \(O\) is either empty or the union of balls.
2. Every point \(x \in O\) is interior to \(O\).

**Proof.** If \(O\) is empty then the equivalence is immediate. Therefore, assume that \(O\) is not empty.

\(\Rightarrow\). Consider any \(x \in O\). By property 1, there is an \(a \in O\) and an \(\varepsilon_a > 0\) such that \(x \in N_{\varepsilon_a}(a) \subseteq O\). I need to show that there is an \(\varepsilon > 0\) such that \(N_{\varepsilon}(x) \subseteq O\). Let \(\varepsilon = \varepsilon_a - d(a, x)\). Consider any \(b \in N_{\varepsilon}(x)\). By the triangle inequality, \(d(b, a) \leq d(b, x) + d(x, a) < \varepsilon_a\). Hence \(b \in N_{\varepsilon_a}(a)\), which implies that \(b \in O\), which implies that \(N_{\varepsilon}(x) \subseteq O\), as was to be shown.

\(\Leftarrow\). For each \(x \in O\), choose \(\varepsilon_x > 0\) such that \(N_{\varepsilon_x}(x) \subseteq O\). Then

\[
O = \bigcup_{x \in O} N_{\varepsilon_x}(x),
\]

as was to be shown. \(\blacksquare\)

Any set that satisfies either of the properties in Theorem 12, and hence both of these properties, is called open. Formally, I define openness using property 1.
Definition 5. Let \((X,d)\) be a metric space. A set \(O \subseteq X\) is open iff it is either empty or the union of balls.

Example 8. In \(\mathbb{R}\), any interval \((a,b)\) is a ball and therefore is open: set \(\varepsilon = (b-a)/2\) and \(x = (a+b)/2\); then \((a,b) = N_\varepsilon(x)\). Alternatively, one can appeal to Theorem 12 and check that every \(x \in (a,b)\) is interior: set \(\varepsilon = \min\{x-a,b-x\}\) and verify that \(N_\varepsilon(x) \subseteq (a,b)\). □

Whether a set \(A \subseteq X\) is open is not intrinsic to \(A\) but, rather, depends on the complement, \(A^c = X \setminus A\), which depends on \(X\). If \((X,d)\) is a metric space, \(\hat{X} \subseteq X\), and \(A \subseteq \hat{X}\), then it is possible for \(A\) to be open in the metric space defined by \(\hat{X}\) (with the metric inherited from \(X\)) even if it is not open in \(X\).

Example 9. Let \(A = [0,1)\).

- If \(X = \mathbb{R}\) then \(A\) is not open. In particular, 0 is not interior to \(A\) for \(X = \mathbb{R}\) since, for any \(\varepsilon > 0\), \(N_\varepsilon(0) = (-\varepsilon,\varepsilon) \not\subseteq [0,1)\).

- If \(X = [0,1)\), hence \(X = A\), then \(A = X\) is open. In particular, 0 is now interior since, in the \(X = [0,1)\) metric space, for any \(\varepsilon \in (0,1)\), \(N_\varepsilon(0) = \{x \in [0,1) : d(x,0) < \varepsilon\} = [0,\varepsilon) \subseteq [0,1)\).

□

Example 10. Let \(A = \mathbb{Q}\).

- If \(X = \mathbb{R}\) then \(A\) is not open. In particular, it is not hard to show that for any \(x \in \mathbb{Q}\) and any \(\varepsilon > 0\), \(N_\varepsilon(x)\) must contain irrational numbers and hence cannot be a subset of \(\mathbb{Q}\).

- If \(X = \mathbb{Q}\) (hence \(X = A\)), then \(A\) is open. In particular, any \(x \in \mathbb{Q}\) is now interior since, in the \(X = \mathbb{Q}\) metric space, for any \(\varepsilon > 0\), \(N_\varepsilon(x) = \{q \in \mathbb{Q} : d(q,x) < \varepsilon\} = (x-\varepsilon,x+\varepsilon) \cap \mathbb{Q} \subseteq \mathbb{Q}\).

□

The next result establishes the fundamental properties of open sets.

Theorem 13. Let \((X,d)\) be a metric space.

1. Both \(X\) and \(\emptyset\) are open.

2. For any two open sets \(O_1\) and \(O_2\), \(O_1 \cap O_2\) is open. By induction, finite intersections of open sets are open.

3. For any set \(O\) of open sets, 
\[
\bigcup_{O \in O} O
\]

is open. Thus, arbitrary unions of open sets are open.
Proof.

1. $X$ is open since it is the union of all balls. $\emptyset$ is open by definition.

2. Let $O_1$ and $O_2$ be open. If $O_1 \cap O_2 = \emptyset$ then $O_1 \cap O_2$ is open by definition. Otherwise, let $x \in O_1 \cap O_2$. Since $x \in O_1$ and $O_1$ is open there is an $\varepsilon_1 > 0$ such that $N_{\varepsilon_1}(x) \subseteq O_1$ (Theorem 12). Similarly, there is an $\varepsilon_2 > 0$ such that $N_{\varepsilon_2}(x) \subseteq O_2$. Set $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Then $N_\varepsilon(x) \subseteq O_1 \cap O_2$.

3. If the $O$ are open then $\bigcup_{O \in \mathcal{O}} O$ is a union of unions of balls, and is therefore open by definition.

The next example shows that arbitrary intersections of open sets need not be open.

Example 11. Suppose $\mathcal{O}$ consists of intervals of the form $(0, 1+1/t)$ for $t \in \{1, 2, \ldots\}$. Then

$$\bigcap_{O \in \mathcal{O}} O = (0, 1],$$

which is not open. □

A metric space is a special case of a more general type of space called a topological space. A topological space specifies a set of points $X$ and a set of open sets $\tau$ (for ‘topology’), with the requirement that the sets in $\tau$ must satisfy the conditions in Theorem 13. The twist is that the sets in $\tau$ don’t have to be derived from a metric and in fact there are topological spaces that do not correspond to any metric space. For example, let $X = \mathbb{R}$ and let $\tau = \{X, \emptyset\}$: the only open sets are the space itself and the empty set. There is no metric on $\mathbb{R}$ that that can generate this $\tau$. The main result on existence of optima generalizes to arbitrary topological spaces. Of the other results, some generalize and some do not.

### 5.2 Open sets and sequences.

**Theorem 14.** Let $(X,d)$ be a metric space, let $(x_t)$ be a sequence in $X$ and let $x^* \in X$. $x_t \to x^*$ iff for any open set $O$ containing $x^*$, there is a $T \in \mathbb{N}$ such that for all $t > T$, $x_t \in O$.

**Proof.** $\Rightarrow$. Let $O$ be an open set containing $x^*$. Since $O$ is open, $x^*$ is interior, and hence there is an $\varepsilon > 0$ such that $N_\varepsilon(x^*) \subseteq O$. Since $x_t \to x^*$, there is a $T$ such that for all $t > T$, $x_t \in N_\varepsilon(x^*) \subseteq O$, which is what I needed to show.

$\Leftarrow$. The claim is immediate since $N_\varepsilon(x^*)$ is open for any $\varepsilon > 0$. ■
5.3 Limit points.

**Theorem 15.** Let \((X, d)\) be a metric space, let \(A \subseteq X\), and let \(x \in X\). The following are equivalent.

1. For any \(\varepsilon > 0\) there is an \(a \in A \cap N_\varepsilon(x)\) such that \(a \neq x\).

2. For any \(\varepsilon > 0\), \(A \cap N_\varepsilon(x)\) is infinite.

3. There is a sequence \((a_t)\) in \(A\) such that \(a_t \to x\) and for all \(s, t\), \(a_s \neq a_t\) and \(a_t \neq x\).

**Proof.**

1. (1) \(\Rightarrow\) (2). By contraposition. Suppose that there is an \(\varepsilon > 0\) such that \(A \cap N_\varepsilon(x)\) is finite. If \(A \cap N_\varepsilon(x) \setminus \{x\}\) is either empty, then (1) is violated. Therefore, suppose that \(A \cap N_\varepsilon(x) \setminus \{x\}\) equals a non-empty finite set \(B = \{a_1, \ldots, a_K\}\). Let \(\delta = \min_{a_k \in B} d(a_k, x)\). Then \(\delta > 0\), since \(B\) is finite and \(a_k \neq x\) for all \(k\). But then \(A \cap N_\delta(x) \setminus \{x\}\) is empty and (1) is violated.

2. (2) \(\Rightarrow\) (3). Choose \(a_1\) to be any point in \(A \cap N_1(x) \setminus \{x\}\). Since \(A \cap N_1(x)\) is infinite, this is must be possible. In general, for any \(t\), choose \(a_t\) to be any point in \(A \cap N_1/t(x) \setminus \{x\}\) that is not equal to any \(a_s\), \(s < t\). Since \(A \cap N_1/t(x)\) is infinite, this must be possible. The sequence \((a_t)\) has the desired properties.

3. (3) \(\Rightarrow\) (1). From (3), there is a sequence \((a_t)\) in \(A\) such that for any \(\varepsilon > 0\) there is a \(T\) such that for all \(t > T\), \(a_t \in A \cap N_\varepsilon(x) \setminus \{x\}\). Hence (1) is satisfied.

A point \(x \in X\) is said to be a limit point of a set \(A \subseteq X\) iff it satisfies any of the properties in Theorem 15, and hence all of the properties in Theorem 15. Formally, I define limit point using property 1.

**Definition 6.** Let \((X, d)\) be a metric space and let \(A \subseteq X\).

1. \(x \in X\) is a limit point of \(A\) iff for any \(\varepsilon > 0\) there is an \(a \in A \cap N_\varepsilon(x)\) such that \(a \neq x\).

2. \(x \in X\) is an isolated point of \(A\) iff \(x \in A\) and \(x\) is not a limit point of \(A\).

**Remark 6.** Note that by Theorem 15 property 2, a set has to be infinite in order to have any limit points. Thus, for example, the set \(A = \{0\}\) has no limit points.

Remember that limits are for sequences. Limit points are for sets. The concepts are related but distinct. Theorem 15 shows that a limit point of a set is the limit of a sequence drawn from that set. The next result establishes a partial converse.
Theorem 16. Let \((X, d)\) be a metric space. Let \((x_t)\) be a sequence in \(X\) and let \(A\) be the range of \((x_t)\). If \(x_t \to x\), then either \(x\) is a limit point of \(A\) or \(A\) is finite and \(x \in A\).

**Proof.** Suppose that \(x_t \to x\) and suppose that \(x\) is not a limit point of \(A\). Then there is an \(\varepsilon > 0\) such that \(N_\varepsilon(x) \cap A\) is either empty or contains only \(x\). Since \(x_t \to x\), this implies that there is a \(T\) such that for all \(t > T\), \(x_t = x\), in which case \(A\) is finite and \(x \in A\). ■

Here are some examples that may help clarify some of these concepts.

**Example 12.** \(0\) is a limit point of both \([0, 1]\) and \((0, 1)\). □

**Example 13.** Let \((x_t)\) be the sequence \((1, 1/2, 1/3, \ldots)\) and let \(A\) be the range of this sequence. Then \(x_t \to 0\). \(0\) is the only limit point of \(A\). Every point of \(A\) is isolated.

**Example 14.** Let \((x_t)\) be the sequence \((0, 0, 0, \ldots)\) and let \(A\) be the range of this sequence: \(A = \{0\}\). Then \(x_t \to 0\). \(A\) has no limit points. □

**Example 15.** \(A = [0, 1]\). Then \(0\) is a limit point of \(A\), but \(A\), which is uncountable, is not the range of a sequence (which must be countable). □

**Remark 7.** If \((x_t)\) is a sequence in \(X\) then \(x \in X\) is an accumulation point or cluster point of \((x_t)\) iff for any \(\varepsilon > 0\) there are infinitely many \(t\) such that \(x_t \in N_\varepsilon(x)\). It is almost immediate that \(x\) is an accumulation point of \((x_t)\) iff there is a subsequence of \((x_t)\) converging to \(x\): “accumulation point” is just another way to say “subsequential limit.”

Let \(A\) be the range of \((x_t)\). If \(x\) is a limit point of \(A\) then \(x\) is an accumulation point of \((x_t)\) and, moreover, there is a subsequence of \((x_t)\) that converges to \(x\) and in which all terms have different values.

On the other hand, an accumulation point of \((x_t)\) need not be a limit point of \(A\). For example, if \((x_t) = (1, 1, 1, \ldots)\) then \(A = \{1\}\), which has no limit points, but \(x = 1\) is an accumulation point of \((x_t)\). □

### 5.4 Closed sets.

**Theorem 17.** Let \((X, d)\) be a metric space and consider any \(C \subseteq X\). The following are equivalent.

1. For any sequence \((x_t)\) in \(C\), if \(x_t \to x\) then \(x \in C\).
2. For any \(x \in X\), if \(x\) is a limit point of \(C\) then \(x \in C\).
3. \(C^c\) is open.

**Proof.**
1. (1) ⇒ (2). Let \( x \) be a limit point of \( C \). By Theorem 15, there is a sequence \((x_t)\) in \( C \) such that \( x_t \to x \). By (1), \( x \in C \).

2. (2) ⇒ (3). By contraposition. Suppose that (3) fails. Then \( C^c \) is not open. Then there exists an \( x \in C^c \) such that \( x \) is not interior to \( C^c \). Hence for any \( \varepsilon > 0 \), \( N_\varepsilon(x) \not\subseteq C^c \), hence \( N_\varepsilon(x) \cap C \neq \emptyset \). Take any \( a \in N_\varepsilon(x) \cap C \). Then \( a \neq x \) since \( x \in C^c \). Since \( \varepsilon \) was arbitrary, this shows that \( x \) is a limit point of \( C \). But since \( x \notin C \), (2) fails.

3. (3) ⇒ (1). By contraposition. Suppose that (1) fails. Then there is a sequence \((x_t)\) in \( C \) and an \( x \in X \) such that \( x_t \to x \) but \( x \notin C \). Since \( x \notin C \), \( x \in C^c \). Take any \( \varepsilon > 0 \). Since \( x_t \to x \), there is an \( x_t \in N_\varepsilon(x) \). Since \( x_t \in C \), hence \( x_t \notin C^c \), \( N_\varepsilon(x) \not\subseteq C^c \). Since \( \varepsilon \) was arbitrary, this shows that \( x \) is not interior to \( C \), hence \( C^c \) is not open, hence (3) fails.

Any set satisfying any of the properties in Theorem 17, hence satisfying all of them, is called closed. Formally, I define closed using property 3.

**Definition 7.** Let \((X,d)\) be a metric space. A set \( C \subseteq X \) is closed iff its complement, \( C^c \), is open.

The next result is the analog of Theorem 13.

**Theorem 18.** Let \((X,d)\) be a metric space.

1. Both \( X \) and \( \emptyset \) are closed.

2. For any two closed sets \( C_1 \) and \( C_2 \), \( C_1 \cup C_2 \) is closed. By induction, finite unions of closed sets are closed.

3. For any set \( C \) of closed sets

\[
\bigcap_{C \in C} C
\]

is closed. Thus, arbitrary intersections of closed sets are closed.

**Proof.** Immediate from Theorem 13 and DeMorgan’s laws (see the Set Theory notes).

The next example shows that arbitrary unions of closed sets need not be closed.

**Example 16.** Suppose \( O \) consists of intervals of the form \([0, 1-1/t]\) for \( t \in \{1, 2, \ldots \} \). Then

\[
\bigcup_{O \in \mathcal{O}} O = [0, 1),
\]

which is not closed. □
The terms “open” and “closed” can be misleading in the following respect. In English, a door, say, can be either open, closed, or partly open, but not both open and closed. This is not true for open and closed for sets. We have already seen that, in a metric space \((X,d)\), both \(X\) and \(\emptyset\) are both open and closed.

A related point is that in \(\mathbb{R}^N\), \(\mathbb{R}^N\) and \(\emptyset\) are the only sets that are both closed and open. But there are other spaces in which this is not true. For example, consider the space \(X = \mathbb{R} \setminus \{0\}\) with the standard Euclidean metric. In this space, \((-\infty, 0)\) is both open and closed, as is \((0, \infty)\). This \(X\) violates a property called connectedness. It is, in fact, the case that for any metric space \(X\), \(X\) is connected iff \(X\) and \(\emptyset\) are the only sets that are both open and closed. See the notes on Connected Spaces.

Another subtlety is that the property of a set being closed is not intrinsic to the set but also depends on the ambient space \(X\), as illustrated by the next example; see also the related discussion of open sets and Example 9.

**Example 17.** Let \(C = [0, 1)\).

- If \(X = \mathbb{R}\) then \(C\) is not closed, since 1 is a limit point of \(C\) but 1 \(\notin C\).
- But if \(X = [0, 1)\), so that \(C = X\), then \(C\) is closed (remember that \(X\) is always closed). In effect, if \(X = [0, 1)\) then the number 1 does not exist.

\(\Box\)

In contrast to the property of being closed, the property of being complete is intrinsic to the set; it does not depend on the ambient space. The next result establishes that a complete set is always closed, regardless of the ambient space. The converse is not true, as illustrated by the previous example: whether \(C = [0, 1)\) is closed depends on whether the space is \(X = [0, 1)\) or \(X = \mathbb{R}\), but either way, \(C\) is not complete.

**Theorem 19.** Let \((X,d)\) be a metric space and let \(C \subseteq X\).

1. If \(C\) is complete then \(C\) is closed.

2. If \(X\) is complete and \(C\) is closed then \(C\) is complete.

**Proof.**

1. Let \((x_t)\) be a sequence in \(C\) and suppose that there is an \(x \in X\) such that \(x_t \to x\). Since \(x_t \to x\), \((x_t)\) is Cauchy (Theorem 9). Since \(C\) is complete, there is an \(x^* \in C\) such that \(x_t \to x^*\). Since also \(x_t \to x\), \(x = x^*\) (Theorem 7), and hence \(x \in C\), hence \(C\) is closed.

2. Take any Cauchy sequence \((x_t)\) in \(C\). Since \(C \subseteq X\), \((x_t)\) is a Cauchy sequence in \(X\). Since \(X\) is complete, there is an \(x \in X\) such that \(x_t \to x\). Since \(C\) is closed, \(x \in C\). Hence \(C\) is complete.
In practice, we will be working mainly with spaces $X$ that are complete, in which case, by the second part of Theorem 19, the statement that $C \subseteq X$ is closed is equivalent to the statement that $C$ is complete; but to take advantage of this equivalence, we may first have to prove that $X$ is complete.

Given a set $A$, the closure of $A$ is the smallest closed set containing $A$.

**Definition 8.** Given a set $A \subseteq X$, $\overline{A}$, called the closure of $A$, is the intersection of all closed sets containing $A$.

This definition is not vacuous since $X$ itself is a closed set containing $A$. As an intersection of closed sets, $\overline{A}$ is closed. It is immediate from the definition that $A \subseteq \overline{A}$.

Given a set $A$, let $A'$ denote the set of limit points of $A$.

**Theorem 20.** Let $(X, d)$ be a metric space. For any set $A \subseteq X$,

$$\overline{A} = A \cup A'.$$

**Proof.** Since $\overline{A}$ is closed and $A \subseteq \overline{A}$, it follows from Theorem 17 that $A \cup A' \subseteq \overline{A}$.

Since $A \subseteq A \cup A'$, it suffices to prove that $A \cup A'$ is closed, since then $\overline{A} \subseteq A \cup A' = A \cup A'$ and hence $\overline{A} = A \cup A'$.

To show that $A \cup A'$ is closed, I argue by contraposition. Consider any $x \notin A \cup A'$. Therefore, $x \notin A$ and $x$ is not a limit point of $A$. Hence there is an $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq A^c$. Consider any point $b \in N_\varepsilon(x)$. Then $b \notin A$. Also, since $N_\varepsilon(x)$ is open, $b$ is interior to $N_\varepsilon(x)$, hence there is an $\varepsilon_b$ such that $N_{\varepsilon_b}(b) \subseteq N_\varepsilon(x) \subseteq A^c$. This implies that $b \notin A'$. Therefore, $b \notin A \cup A'$. Since this holds for any $b \in N_\varepsilon(x)$, $x$ is not a limit point of $A \cup A'$. In summary, if $x$ is not in $A \cup A'$ then it is not a limit point of $A \cup A'$. By contraposition, of $x$ is a limit point of $A \cup A'$ then $x \in A \cup A'$, as was to be shown. ■

In $\mathbb{R}$, finite unions of closed intervals are closed, but closed sets in $\mathbb{R}$ can be much more complicated than finite unions of closed intervals.

**Example 18.** Let $E_0 = [0, 1] \setminus (1/3, 2/3) = [0, 1/3] \cup [2/3, 1]$, let $E_1 = E_0 \setminus ((1/9, 2/9) \cup (7/9, 8/9)) = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$ and so on. In words, at each stage, remove the (open) middle third from each remaining subinterval. Let $A = \bigcap E_t$. $A$ is called the *Cantor set*. $A$ is closed, since it is an intersection of closed sets. In fact, one can show that $A$ is *perfect*, meaning that $A = A'$. And $A$ is uncountable.\(^4\) But $A^c$ is open (since $A$ is

\(^4\)This is true in general of perfect sets in $\mathbb{R}$. But it can be seen as follows. Write the elements of $[0, 1]$ as ternary expansions (i.e., in the form $x = x_1/3 + x_2/9 + x_3/27 + \ldots$) and then note that $x \in A$ iff it has a ternary expansion with no 1s in it. Thus $x \in A$ iff it has a ternary expansion that is a string of 0s and 2s. Writing a 1 in place of a 2 puts the set of such strings into 1-1 correspondence with the set of strings of 0s and 1s, which can be identified with the set of binary expansions of elements of $[0, 1]$.}
closed) and one can show that $A^c$ is dense in $[0,1]$, meaning that $\overline{A^c} = [0,1]$. So $A$ is large in one sense (it is uncountable) but small in another (it is the complement of a set that is open and dense). □

Intuitively, the closed $\varepsilon$ ball, $N_\varepsilon(x)$, should equal $\{a \in X : d(a,x) \leq \varepsilon\}$. This turns out to be true in normed vector spaces but not in general.

**Theorem 21.** Let $(X,d)$ be a metric space. For any $x \in X$ and any $\varepsilon > 0$, $N_\varepsilon(x) \subseteq \{a \in X : d(a,x) \leq \varepsilon\}$.

**Proof.** By Theorem 20, since $N_\varepsilon(x) \subseteq \{a \in X : d(a,x) \leq \varepsilon\}$, it remains to show if $a$ is a limit point of $N_\varepsilon(x)$, then $d(a,x) \leq \varepsilon$. Equivalently, I need to argue that if $(a_t)$ is a sequence in $N_\varepsilon(x)$ and $a_t \rightarrow a$ then $d(a,x) \leq \varepsilon$. By the triangle inequality, since $a_t \in N_\varepsilon(x)$, $d(a,x) \leq d(a,a_t) + d(a_t,x) < d(a,a_t) + \varepsilon$. Since $a_t \rightarrow a$, $d(a,a_t)$ can be made arbitrarily small, which implies $d(a,x) \leq \varepsilon$. ■

A corollary of Theorem 21 is that the closure of a bounded set is bounded.

**Theorem 22.** Let $(X,d)$ be a metric space. If $B \subseteq X$ is bounded then $\overline{B}$ is also bounded.

**Proof.** Since $B$ is bounded, there is an $x \in X$ and an $r > 0$ such that $B \subseteq N_r(x)$, hence $\overline{B} \subseteq \overline{N_r(x)} \subseteq \{a \in X : d(a,x) \leq r\} \subseteq N_{r+1}(x)$. ■

In metric spaces that are not normed vector spaces, the set inclusion $\overline{N_\varepsilon(x)} \subseteq \{a \in X : d(a,x) \leq \varepsilon\}$ can be strict.

**Example 19.** Consider $(X,d_d)$ where $d_d$, the discrete metric, was defined in Section 1.3.3. For any $x \in X$, $N_1(x) = \{x\}$, $\overline{N_1(x)} = \{x\}$, but $\{a \in X : d(a,x) \leq 1\} = X$. □

**Theorem 23.** Let $X$ be a vector space with norm $\|\cdot\|$ and associated metric $d$. For any $x \in X$ and any $\varepsilon > 0$, $N_\varepsilon(x) = \{a \in X : d(a,x) \leq \varepsilon\}$.

**Proof.** For ease of notation, take $x = 0$. Take any $\varepsilon > 0$. By Theorem 21, it suffices to show that if $\|a\| = \varepsilon$, then $a$ is a limit point of $N_\varepsilon(0)$. Since $a \notin N_\varepsilon(0)$, this means showing that for any $\delta > 0$, $N_\varepsilon(0) \cap N_\delta(a) = \emptyset$. It suffices to restrict attention to $\delta$ that is small, specifically $\delta < \varepsilon$. Take any $\gamma \in (1-\delta/\varepsilon,1)$. Since $\gamma < 1$, $\gamma a \in N_\varepsilon(0)$. And $d(a,\gamma a) = \|a - \gamma a\| = (1-\gamma)\|a\| = (1-\gamma)\varepsilon < \delta$ (since $\gamma > 1 - \delta/\varepsilon$), hence $\gamma a \in N_\delta(a)$. ■

Finally, I note that in $\mathbb{R}$, the inf and sup of a bounded set that is closed must be in the set.

**Theorem 24.** Let $C \subseteq \mathbb{R}$ be closed. If $C$ is bounded above, then $\sup C \in C$. Similarly, if $C$ is bounded below, then $\inf C \in C$.  

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Proof. If \( C \) is bounded above then by the LUB property, \( C \) has a least upper bound, say \( b \in \mathbb{R} \). Since \( b \) is the least upper bound, for any \( t \in \{1, 2, \ldots \} \), \( b - 1/t \) is not an upper bound, and hence there is an \( x_t \in C \) such that \( x_t > b - 1/t \). Since \( b \) is an upper bound, \( x_t \leq b \). For the sequence \( (x_t) \) thus formed, for any \( t \geq T \), \( x_t \in N_{1/T}(b) \). Thus \( x_t \rightarrow b \). Since \( C \) is closed, \( b \in C \).

The proof for \( C \) bounded below is almost identical. ■

6 Strongly Equivalent Metrics.

Definition 9. Let \( X \) be a set. Two metrics \( d_1 \) and \( d_2 \) on \( X \) are strongly equivalent if there exist real numbers \( 0 < a \leq b \) such that for any \( x, y \in X \),
\[
    ad_2(x, y) \leq d_1(x, y) \leq bd_2(x, y).
\]
(1)

Recall that the notes on Vector Spaces and Norms introduced the very similar concept of equivalent norms.

Theorem 25. Let \( X \) be a vector space, let \( f_1 \) and \( f_2 \) be norms on \( X \), and let \( d_1 \) and \( d_2 \) be the associated induced metrics. Then \( f_1 \) and \( f_2 \) are equivalent iff \( d_1 \) and \( d_2 \) are strongly equivalent.

Proof. Almost immediate from the definitions. ■

One implication is that in any finite dimensional vector space, all norm-based metrics are strongly equivalent, since (as discussed in the notes on Vector Spaces and Norms and proved in the notes on Existence of Optima) all norms on a finite dimensional vector space are equivalent.

The main result of this section, Theorem 27, is that if two metrics are strongly equivalent then they have the same bounded sets, totally bounded sets, convergent sequences, Cauchy sequences, open sets, closed sets, and complete sets.

I begin by noting, in Theorem 26, that if two metrics on \( X \) are strongly equivalent, then, around any given point \( x^* \in X \), they generate open balls that are nested, like matryoshka dolls. (I made a similar observation in the notes on Vector Spaces.)

As notation, if \( d_1 \) and \( d_2 \) are two metrics on \( X \), then, for any \( \varepsilon > 0 \) and any \( x^* \in X \), let \( N_\varepsilon(x^*)|_1 \) denote the open \( \varepsilon \) ball centered on \( x^* \) under the \( d_1 \) metric, and let \( N_\varepsilon(x^*)|_2 \) denote the open \( \varepsilon \) ball centered on \( x^* \) under the \( d_2 \) metric.

Theorem 26. Let \( d_1 \) and \( d_2 \) be strongly equivalent metrics on \( X \). Then for any \( \varepsilon_1 > 0 \) there are \( \hat{\varepsilon}_2 \geq \varepsilon_2 > 0 \) such that for any \( x^* \),
\[
    N_{\hat{\varepsilon}_2}(x^*)|_2 \subseteq N_{\varepsilon_1}(x^*)|_1 \subseteq N_{\hat{\varepsilon}_2}(x^*)|_2.
\]
(2)

Conversely, for any \( \varepsilon_2 > 0 \) there are \( \hat{\varepsilon}_1 \geq \varepsilon_1 > 0 \) such that for any \( x^* \),
\[
    N_{\varepsilon_1}(x^*)|_1 \subseteq N_{\varepsilon_2}(x^*)|_2 \subseteq N_{\varepsilon_1}(x^*)|_1.
\]
(3)
Proof. Since $d_1$ and $d_2$ are strongly equivalent, there exist $0 < a \leq b$ such that for any $x, y \in X$, Inequality 1 holds.

1. Subset Relation 2. Fix $\varepsilon_1 > 0$ and let $\varepsilon_2 = \varepsilon_1/b$ and $\hat{\varepsilon}_2 = \varepsilon_1/a$; since $b \geq a > 0$, these are well defined with $\hat{\varepsilon}_2 \geq \varepsilon_2$. Take any $x^*$.

   • To see that $N_{\varepsilon_2}(x^*)|_2 \subseteq N_{\hat{\varepsilon}_2}(x^*)|_1$, consider any $x \in N_{\varepsilon_2}(x^*)|_2$. Then $d_2(x, x^*) < \varepsilon_2 = \varepsilon_1/b$, which implies $bd_2(x, x^*) < \varepsilon_1$. By Inequality 1, this implies $d_1(x, x^*) < \varepsilon_1$, which implies $x \in N_{\hat{\varepsilon}_2}(x^*)|_1$, which proves $N_{\varepsilon_2}(x^*)|_2 \subseteq N_{\hat{\varepsilon}_2}(x^*)|_1$.

   • To see that $N_{\hat{\varepsilon}_1}(x^*)|_1 \subseteq N_{\varepsilon_2}(x^*)|_2$, consider any $x \in N_{\hat{\varepsilon}_1}(x^*)|_1$. Then $d_1(x, x^*) < \hat{\varepsilon}_1$. By inequality 1, this implies $ad_2(x, x^*) < \varepsilon_1$, which implies $d_2(x, x^*) < \varepsilon_1/a = \varepsilon_2$. Hence $x \in N_{\varepsilon_2}(x^*)|_2$, which proves $N_{\hat{\varepsilon}_1}(x^*)|_1 \subseteq N_{\varepsilon_2}(x^*)|_2$.

2. Subset Relation 3. Fix $\varepsilon_2 > 0$ and set $\varepsilon_1 = a\varepsilon_2$ and $\hat{\varepsilon}_1 = b\varepsilon_2$. The rest of the argument is so similar that I omit it.

\[\Box\]

**Theorem 27.** Let $X$ be a set and let $d_1$ and $d_2$ be strongly equivalent metrics on $X$.

1. For any $A \subseteq X$, $A$ is bounded under $d_1$ iff it is bounded under $d_2$.

2. For any $A \subseteq X$, $A$ is totally bounded under $d_1$ iff it is totally bounded under $d_2$.

3. For any sequence $(x_i)$ and point $x^*$ in $X$, $(x_i)$ converges to $x^*$ under $d_1$ iff it converges to $x^*$ under $d_2$.

4. For any sequence $(x_i)$ in $X$, $(x_i)$ is Cauchy under $d_1$ iff it is Cauchy under $d_2$.

5. For any $A \subseteq X$, $A$ is open under $d_1$ iff it is open under $d_2$.

6. For any $A \subseteq X$, $A$ is closed under $d_1$ iff it is closed under $d_2$.

7. For any $A \subseteq X$, $A$ is complete under $d_1$ iff it is complete under $d_2$.

**Proof.** For each iff claim, the iff proof is the same in both directions apart from labeling and so I provide only one direction.

1. Boundedness. If $A$ is $d_1$ bounded then there is an $r_1 > 0$ and an $x^* \in X$ such that $A \subseteq N_{r_1}(x^*)|_1$. By Theorem 26 (Subset Relation 2), there is an $r_2 > 0$ such that $N_{r_1}(x^*)|_1 \subseteq N_{r_2}(x^*)|_2$, which shows that $A$ is $d_2$ bounded.
2. Total Boundedness. Suppose that \( A \) is \( d_1 \) totally bounded. Take any \( \varepsilon_2 > 0 \) and let \( \varepsilon_1 > 0 \) be as in Theorem 26 (Subset Relation 3). Since \( A \) is \( d_1 \) totally bounded there is a finite set \( E \) such that \( A \subseteq \bigcup_{x \in E} N_{\varepsilon_1}(x)|_1 \). For each \( x \in E \), we have \( N_{\varepsilon_1}(x)|_1 \subseteq N_{\varepsilon_2}(x)|_2 \) and so \( A \subseteq \bigcup_{x \in E} N_{\varepsilon_2}(x)|_2 \), which implies that \( A \) is \( d_2 \) totally bounded.

3. Convergence. Suppose that \( (x_t) \) \( d_1 \) converges to \( x^* \). Take any \( \varepsilon_2 > 0 \) and let \( \varepsilon_1 \) be as in Theorem 26 (Subset Relation 3). Since \( (x_t) \) \( d_1 \) converges to \( x^* \), there is a \( T \) such that for all \( t > 0T \), \( d_1(x_t, x^*) < \varepsilon_1 \), hence \( d_2(x_t, x^*) < \varepsilon_2 \), which implies that \( (x_t) \) \( d_2 \) converges to \( x^* \).

4. Cauchy. Suppose that \( (x_t) \) is \( d_1 \) Cauchy. Take any \( \varepsilon_2 > 0 \) and let \( \varepsilon_1 \) be as in Theorem 26 (Subset Relation 3). Since \( (x_t) \) is \( d_1 \) Cauchy, there is a \( T \) such that for all \( s, t > T, d_1(x_s, x_t) < \varepsilon_1 \), which implies \( d_2(x_s, x_t) < \varepsilon_2 \), which implies that \( (x_t) \) is \( d_2 \) Cauchy.

5. Open. Let \( A \) be \( d_1 \) open. Consider any \( x \in A \). Since \( A \) is \( d_1 \) open, \( x \) is \( d_1 \) interior to \( A \): there exists an \( \varepsilon_1 > 0 \) such that \( N_{\varepsilon_1}(x)|_1 \subseteq A \). Let \( \varepsilon_2 > 0 \) be as in Theorem 26 (Subset Relation 2) with \( N_{\varepsilon_2}(x)|_2 \subseteq N_{\varepsilon_1}(x)|_1 \). Then \( x \) is \( d_2 \) interior to \( A \) and hence \( A \) is open.

6. Closed. This follows from the previous result for open sets and the definition of a closed set as the complement of an open set.

7. Complete. Let \( A \) be \( d_1 \) complete. If \( (x_t) \) is \( d_2 \) Cauchy then (by the previous result for Cauchy sequences) \( (x_t) \) is also \( d_1 \) Cauchy, which implies, since \( A \) is \( d_1 \) complete, that there is an \( x^* \in A \) such that \( (x_t) \) \( d_1 \) converges to \( x^* \), which implies (by the previous result for convergence) that \( (x_t) \) \( d_2 \) converges to \( x^* \).

\( \square \)

The proof that, in \( \mathbb{R}^N \), bounded sets are totally bounded in \( \mathbb{R}^N \) follows quickly from Theorem 27.

**Proof of Theorem 4.** In the notes on Vectors Spaces and Norms, I showed that the Euclidean and max norms on \( \mathbb{R}^N \) are equivalent, which implies that their induced metrics are strongly equivalent. In view of Theorem 27, it therefore suffices to prove that, under \( d_{\text{max}} \), bounded sets in \( \mathbb{R}^N \) are totally bounded.

Suppose that \( A \subseteq \mathbb{R}^N \) is bounded under \( d_{\text{max}} \). Then there is an \( r \) such that \( A \) is contained in an open \( r \) ball, and is therefore contained in a closed \( r \)-ball. Under \( d_{\text{max}} \), a closed \( r \) ball is an \( N \)-dimensional cube with edges of length \( 2r \). Fix any \( \varepsilon > 0 \) and take any integer \( T > r/\varepsilon \). Divide the cube into \( T^N \) subcubes, each with edges of length \( 2r/T \). Since \( T > r/\varepsilon \), each subcube is contained in an open \( \varepsilon \) ball (which, under \( d_{\text{max}} \), is an open cube). Hence \( A \) is covered by a finite \( (T^N) \) number
of open $\varepsilon$ balls. $A$ is, therefore, totally bounded. ■

In infinite dimensional vector spaces, different norms may not be equivalent and, as a consequence, the induced metrics may have different convergence properties. The following example illustrates what can go wrong.

*Example 20.* Let $X = \ell^\infty$ and consider the sup and $p^*$ norms, where the latter (first introduced in the notes on Vector Spaces and Norms) is given by

$$
\|x\|_{p^*} = \sup_n \frac{|x_n|}{n}.
$$

Let $d_{\text{sup}}$ and $d_{p^*}$ be the associated metrics. These norms are not equivalent (as already discussed in the notes on Vector Spaces and Norms), and the associated metrics are not strongly equivalent.

Consider the sequence $(x_t)$ with terms $x_1 = (1, 0, 0, \ldots)$, $x_2 = (0, 1, 0, 0, \ldots)$, and so on. Then this sequence converges to the origin under $d_{p^*}$ but does not converge (to anything) under $d_{\text{sup}}$. ■

**References**