Semi-Continuity\footnote{This work is licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 4.0 License.}

1 Definition.

Let \((X, d)\) be a metric space. For a function \(f : X \to \mathbb{R}\) and a point \(y \in \mathbb{R}\), the upper contour set defined by \(y\) is

\[
U(y) = f^{-1}((y, \infty)) = \{x \in X : f(x) \geq y\}.
\]

The lower contour set defined by \(y\) is

\[
L(y) = f^{-1}((\infty, y]) = \{x \in X : f(x) \leq y\}.
\]

The next result establishes that a number of properties are equivalent.

**Theorem 1.** Let \(f : X \to \mathbb{R}\).

1. The following are equivalent.
   \begin{enumerate}[(a)]
   
   
   \item For any \(y \in \mathbb{R}\), \(U(y)\) is closed.
   \item For any \(y \in \mathbb{R}\), \(f^{-1}((\infty, y]) = [U(y)]^c\) is open.
   \item For any \(x \in X\), if the sequence \((x_t)\) in \(X\) converges to \(x\), then for any \(\varepsilon > 0\) there is a \(T\) such that for all \(t > T\), \(f(x) > f(x_t) - \varepsilon\).
   \end{enumerate}

2. The following are equivalent.
   \begin{enumerate}[(a)]
   
   \item For any \(y \in \mathbb{R}\), \(L(y)\) is closed.
   \item For any \(y \in \mathbb{R}\), \(f^{-1}((y, \infty)) = [L(y)]^c\) is open.
   \item For any \(x \in X\), if the sequence \((x_t)\) in \(X\) converges to \(x\) then for any \(\varepsilon > 0\) there is a \(T\) such that for all \(t > T\), \(f(x) < f(x_t) + \varepsilon\).
   \end{enumerate}

**Proof.** I provide the proof of equivalence for the first set of conditions. The proof for the second set of conditions is analogous.

- \(1(a) \Rightarrow 1(b)\). Almost immediate, since \(U(y)\) is closed iff \([U(y)]^c\) is open.
1(b) ⇒ 1(c). By contraposition. Suppose that there is an \( x \in X \) and a sequence \((x_t)\) in \( X \) that converges to \( x \) such that for some \( \varepsilon > 0 \) there are infinitely many \( t \) such that \( f(x) \leq f(x_t) - \varepsilon \). Choose any \( y \in (f(x), f(x) + \varepsilon) \). Then there are infinitely many \( t \) such that \( x_t \in U(y) \). These \( x_t \) constitute a sequence in \( U(y) \) that converges to \( x \), but \( x \notin U(y) \), hence \( U(y) \) is not closed, hence \([U(y)]^c\) is not open.

1(c) ⇒ 1(a). Take any \( y \in \mathbb{R} \). If \( U(y) = \emptyset \) then I am done. Otherwise, take any convergent sequence \((x_t)\) in \( U(y) \) let \( x = \lim x_t \). I need to show that \( x \in U(y) \). By 1(c), for any \( \varepsilon > 0 \) there is a \( T \) such that for all \( t > T \), \( f(x) > f(x_t) - \varepsilon \). Since \( x_t \in U(y) \), \( f(x_t) \geq y \), hence \( f(x) > y - \varepsilon \). Since this must hold for any \( \varepsilon > 0 \), \( f(x) \geq y \), which implies \( x \in U(y) \).

Since conditions listed under 1 and 2 are equivalent, I can choose any pair of them to define upper and lower semicontinuity. To underscore the analogy with continuity, I use the “b” conditions.

**Definition 1.** Let \( f : X \to \mathbb{R} \).

1. \( f \) is upper semicontinuous (USC) iff for any \( y \in \mathbb{R} \), \( f^{-1}((-\infty, y)) \) is open.
2. \( f \) is lower semicontinuous (LSC) iff for any \( y \in \mathbb{R} \), \( f^{-1}((y, \infty)) \) is open.

Informally, a function is upper semicontinuous if it is continuous or, if not, it only jumps up; a function is lower semicontinuous if it is continuous or, if not, it only jumps down.

**Example 1.** Define \( f : \mathbb{R} \to \mathbb{R} \) by

\[
f(x) = \begin{cases} 
1/x & \text{if } x < 0, \\
0 & \text{if } x = 0, \\
-1/x & \text{if } x > 0.
\end{cases}
\]

Then \( f \) is upper semi-continuous. In particular, if \( x_t \to 0 \), then \( f(x_t) \to -\infty < 0 = f(0) \). □

**Theorem 2.** \( f \) is continuous iff it is both upper and lower semi-continuous.

**Proof.** Almost immediate form property 1(c) and 2(c), which together are equivalent to requiring that if \((x_t)\) converges to \( x \) then for any \( \varepsilon > 0 \), there is a \( T \) such that for all \( t > T \), \( f(x_t) \in N_\varepsilon(f(x)) \), hence \( f(x_t) \) converges to \( f(x) \). □