The Tychonoff Theorem for Countable Products of Compact Sets

1 Countable product sets and pointwise convergence.

These notes provide a slight generalization of the version of Tychonoff’s Theorem given in the notes on Completeness and Compactness in $\mathbb{R}^\omega$.

Given a countable (finite or infinite) collection of metric spaces $(X_1, d_1), (X_2, d_2), \ldots$, let $X$ denote the product space $X = X_1 \times X_2 \times \ldots$. Thus $x = (x_1, x_2, \ldots)$ is in $X$ iff $x_n \in X_n$ for each $n$. For infinite products, the leading example is $X_n = \mathbb{R}$ for every $n$, in which case $X = \mathbb{R}^\omega$. But this is just an example: $X_n$ can by any metric space, including, say $(\ell^\infty, d_{\text{sup}})$.

A sequence $(x_t)$ in $X$ converges to $x$ pointwise iff it converges in each coordinate: $x_{tn} \to x_n$ for each $n$, where convergence is with respect to $d_n$. The Tychonoff Theorem for countable products states that if the $X_n$ are all compact then $X$ is compact under pointwise convergence.

The next subsection verifies that there is a metric on $X$ for which convergence is pointwise, but this fact is not needed for the statement and proof of the Tychonoff Theorem, which is why I have isolated the metric discussion to a separate subsection.

2 A metric for pointwise convergence.

Pointwise convergence on a countable product set can be induced by the metric $d_{\text{pw}}$ (“pw” for “pointwise”) defined as

$$d_{\text{pw}}(x, y) = \sup_n \frac{\min\{1, d_n(x_n, y_n)\}}{n},$$

where $x_n, y_n \in X_n$ and $d_n$ is the metric on $X_n$. This $d_{\text{pw}}$ is a generalization of the $d_{\text{pw}}$ metric defined for $\mathbb{R}^\omega$ in the Metric Space notes and discussed further in the notes on Completeness and Compactness in $\mathbb{R}^\omega$.

**Theorem 1.** Let $X$ be a countable product of metric spaces. Then $d_{\text{pw}}$ as defined above is a metric.

**Proof.** Exercise. ■

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**Theorem 2.** Let \( (x_t) \) be a sequence in \((X, d_{pw})\). \( x_t \to x \) iff \( x_{tn} \to x_n \) for all \( n \). \((x_t)\) is Cauchy iff \((x_{tn})\) is Cauchy for all \( n \).

**Proof.** The proof is essentially identical to the one given in the notes on Completeness and Compactness in \( \mathbb{R}^\omega \). ■

3 The Tychonoff Theorem for countable products.

**Theorem 3** (Countable Tychonoff). Let \( X \) be a countable product of compact sets. Then \( X \) is compact under pointwise convergence.

**Proof.** The proof is almost identical to the proof in the notes on Completeness and Convergence in \( \mathbb{R}^\omega \). ■

The general version of the Tychonoff Theorem (see, for example, ?) establishes that an arbitrary (not necessarily countable) product of compact sets is compact in the product topology (I mentioned topologies briefly in the Metric Space notes).