Elementary Endogenous Growth Theory

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A. Development of Endogenous Growth Theory

- Early work prior to 1980:
  - von Neumann (1937) – linear production & balanced growth
  - Solow (1956) – increasing returns & sustained growth
  - Pitchford (1960) – decreasing returns with capital-labor as substitutes & sustained growth
  - Arrow (1962) – learning by doing & growth
  - Uzawa (1965) – human capital, education & growth
  - Nelson and Phelps (1966) – human capital & productivity growth
  - Shell (1966) – inventive activity & growth
  - Arrow (1969) – technology spillover & growth
  - Wan (1970) – learning, innovation & growth

- Pivotal studies of new growth theory:
  - Romer (1986) – general or knowledge capital
  - Stokey (1988) – learning by doing
  - Rebelo (1991) – the AK model
• New stylized facts:

(S1) Club-convergence: bi-mode (Quah 1996)

(S2) Divergent factor accumulation paths:
- US high K high H
- Macao low K high H (Taiwan - H-intensive)
- Congo high K low H (Korea - K-intensive)
- Zaire low K low H

(S3) Increasing rates of growth for the leading economy:
- 1700-1785 Netherlands θ = - 0.07%
- 1785-1820 UK θ = 0.5%
- 1820-1890 UK θ = 1.4%
- 1890-1970 US θ = 2.3%

(S4) Migration of both skilled/unskilled to rich countries: why would the unskilled migrate if paid at their marginal product?

(S5) Over-taking/lagging behind development experiences: why did Korea/Taiwan over-take the Philippines? why did Argentina dropped from the high rank in 1900?

(S6) Cross-country productivity disparities and widened world income differences
- Birth of endogenous growth theory

<table>
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<th>Constant Returns or Long-run constant return</th>
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Note: Those highlighted (in italic) will be discussed in this note – thus, our focus will only be on stylized facts (S1), (S3), (S4) and (S6) for now.
B. One-Sector Endogenous Growth Theory

To allow for sustained growth, the one-sector production must feature constant (our focus) or increasing returns, implying that the marginal product of capital (MPK) is strictly positive even when the capital stock grows to infinity. We will present the basic setup, followed by three useful economic models with different focuses that can be applied to different growth and development issues. For simplicity, we normalize the population to 1, so the population growth rate is zero (n = 0).

1. The Basic Setup

• Lifetime utility: \( U = \int_0^\infty u(c)e^{-\rho t} dt \), with \( u(c) = \frac{c^{1-\sigma^{-1}} - 1}{1 - \sigma^{-1}} \)

• Periodic budget constraint & capital evolution:
  \[ \dot{k} = f(k) - \delta k - c \]
  where the initial capital stock per capita is \( k(0) = k_0 > 0 \)

• Main feature: The setup is similar to the Ramsey optimal growth model except that the production function \( f(k) \) now features strictly positive MPK (e.g., \( f(k) = AK \) or \( f(k) = AK+BK^\alpha \))
• In competitive markets, capital efficiency is reached when MPK equals the market rental rate $r$:
  \[ f_k = r \text{ (KE locus)} \]
  which is drawn under constant returns ($MPK = \text{constant}$)
• The Keynes-Ramsey equation:
  \[ \dot{\theta} = \frac{\dot{c}}{c} = \sigma[f_k - (\rho + \delta)] \]
  which, with the use of (KE), can be rewritten as,
  \[ r = (\rho + \delta) + \sigma^{-1} \theta \text{ (KR)} \]
• Endogenous vs. exogenous growth:
  o the KR locus is the same in both exogenous and endogenous growth models
  o $\theta$ is exogenous ($= \eta$) in the exogenous growth model
  o the endogenous balanced growth rate ($\theta$) is determined in the endogenous growth model by the KE and KR loci jointly, where under constant returns, $c$, $k$ and $y$ all grow at this common rate $\theta$
2. The AK Model: Rebelo (1988)

- **Production function:** \( y = f(k) = Ak \)
- **Endogenous growth rate:** \( \theta = \sigma[A - (\rho + \delta)] > 0 \)
  - higher productivity \( A \) => MPK↑ => encourage capital accumulation & raise growth
  - higher intertemporal substitution σ or lower time discounting ρ => encourage saving => raise capital and output growth
- **Main implications:**
  - Countries with different \( A \) will grow at different growth rates, implying non-convergence (S1)
    - problem: why China, with low \( A \), has grown so rapidly?
  - Any policy affecting the level of \( A \) has a growth effect
    - example: standing on the giant’s shoulder effect
Appendix: General Production

- \( Y = AK + BK^\alpha \) (Sobelo production function)
- \( MPK = A + \alpha BK^{\alpha-1} \Rightarrow \lim_{K \to \infty} MPK = A \)
- Consider:
  - Non-degenerate equilibrium: \( A > \rho \)
  - \( \sigma^{-1} > 1 \)
- KR: \( r = \rho + \sigma^{-1} \theta \)
- KE:
  \[
  \frac{\dot{Y}}{Y} = \frac{AK\dot{K}}{Y} + \frac{\alpha BK^\alpha \dot{K}}{YK} \\
  \frac{\dot{Y}}{Y} = \frac{AK}{K} + \frac{\alpha BK^\alpha}{K} \\
  \Rightarrow \\
  \frac{\dot{Y}}{Y} \theta = \theta_K (A + \alpha BK^\alpha) \\
  = \theta_K \cdot r
  \]

- Production function:  
  \[ y = f(k, \bar{K}) = Ak^{1-\beta}\bar{K}^\beta \]
  - a firm’s output depends on its own capital and the society’s aggregate capital \( \bar{K} \)
  - higher aggregate capital raises individual output as a result of \textit{uncompensated knowledge spillover}
  - to each individual, \( \bar{K} \) is taken as given
  - in equilibrium, \( \bar{K} = k \) (recall that population = 1)

- Remarks:
  - \( \bar{K} \) includes physical, human, knowledge and firm organizational capital
  - Externality (or the spillover) causes free-rider problem and results in underinvestment

- Capital efficiency:  
  \[ r = \frac{MPK}{K} = A(1-\beta)\left(\frac{\bar{K}}{k}\right)^\beta - \delta = A(1-\beta) - \delta > 0 \]

- Endogenous growth rate:  
  \[ \theta = \sigma[A(1-\beta) - (\rho + \delta)] > 0 \]
  - \( \theta \) is increasing in \( A \) and \( \sigma \) but decreasing in \( \rho \), the same as in the AK-model
- the new insight:
  - the stronger the uncompensated knowledge spillover ($\beta$) is, the lower the endogenous growth rate will be
  - intuition: the free-rider problem leads to under-investment
  - with no free-rider problem ($\beta = 0$), it reduces to the AK-model
- Policies to correct the underinvestment problem
  - Free education: education subsidy
  - Patent protection policy: provide public subsidy to inventors but do not prohibit the use of the invented knowledge
  - Public provision of investment incentives: investment-tax credit, successful cases found in East Asian countries


- Setup: $y = f(k) = Ak^{1-\beta} G^\beta$
- Public capital $G = \tau y$ in equilibrium; each individual takes $G$ as given
- 3 types of public provision of capital
  - Infrastructure: highways, satellite
  - Public provision of education or knowledge
  - Public provision of physical capital: science park, utility, etc.
• Endogenous balanced growth rate:
\[ \theta = \sigma \left[ (1 - \beta)A^{1/(1-\beta)} \tau^{\beta/(1-\beta)} (1 - \tau) - (\rho + \delta) \right] \]
which now depends on \( \tau \)

• Growth-maximizing flat tax rate: \( \tau^* = \beta \)
  - For \( \tau^* < \beta \), higher tax rate corrects free rider problem and raises growth
  - For \( \tau^* > \beta \), higher tax rate reduces investment incentive and growth
  - This yields the modified Laffer curve
  - To maximize economic growth, the tax rate should be set the productive public good’s share (\( \beta \))

• Growth-maximizing government size: \( G/y = \tau^* = \beta \)
  - Empirical observations: ambiguous cross-country relationships between government size and growth.

• Remarks: the conventional Laffer curve that defines the relationship between total tax revenue and the tax rate
C. Multi-Sector Endogenous Growth Theory

While the one-sector model offers clean setups and useful explanations for (S1) and (S2), it is ultimately too stylized to capture many other real world observations. We will therefore shift our attention to two much more realistic multi-sector frameworks, one with endogenous human capital accumulation and one with endogenous technical progress.

1. Human Capital Based Endogenous Growth Model

- Uzawa (1965): the rate at which human capital is accumulated depends positively on both the stock of human capital and the education input
- Nelson and Phelps (1966): if the prevailing technology $A$ is below the frontier technology $\overline{A}$, then the productivity growth rate is increasing in both this technology gap and the stock of human capital $H$
- Lucas (1988) utilizes the Uzawa argument to develop a two-sector endogenous growth model with perpetual physical and human capital accumulation. He begins with a model with constant returns, followed by one with increasing returns in the presence of positive uncompensated human capital spillovers

• By incorporating human capital, the new concept of the labor input is now “effective labor,” defined as the multiple of work hours (u) and human capital (H): \( \text{L}=uH \)

• Lifetime utility: \( \text{U} = \int_{0}^{\infty} \frac{\text{c}^{1-\sigma^{-1}} - 1}{1-\sigma^{-1}} \text{e}^{-\rho t} \text{dt} \) (same as before)

• Goods production: \( \text{F} = \text{AK}^{1-\beta}(uH)^{\beta} \) (constant-returns in the long run)

• Physical capital accumulation (budget constraint):
  \[ \dot{\text{K}} = \text{F}(\text{K}, u\text{H}) - \delta \text{K} - \text{c}, \text{ with } \text{K}(0) = \text{K}_{0} > 0 \]

• Human capital accumulation (linear & hence constant-returns):
  \[ \dot{\text{H}} = \phi(1-u)\text{H}, \text{ with } \text{H}(0) = \text{H}_{0} > 0 \]
  where \( \phi \) measures the maximal human capital growth rate with \( 1-u=1 \)

• The constant-returns technologies with constant intertemporal elasticity of substitution preferences imply:
  \[ \frac{\dot{\text{K}}}{\text{K}} = \frac{\dot{\text{H}}}{\text{H}} = \frac{\dot{\text{Y}}}{\text{Y}} = \frac{\dot{\text{C}}}{\text{C}} = 0, \text{ that is,} \]
  consumption, output and the two production inputs all grow at the common rate \( \theta \) (to be determined endogenously)
• Human capital accumulation thus leads to: \( \theta = \frac{\dot{H}}{H} = \phi(1-u) \)

• Keynes-Ramsey (KR): \( \theta = \frac{\dot{c}}{c} = \sigma[MPK - (\rho + \delta)] \)

• Intertemporal no-arbitrage in K and H (IN): \( MPK - \delta = MPH = \phi \)

• (KR) and (IN) => the endogenous growth rate: \( \theta = \sigma(\phi - \rho) \)

• Main findings:
  o Other than preference parameter \((\sigma, \rho)\), economic growth is driven only by the maximal human capital accumulation rate \(\phi\)
  o Higher \(\phi\) => the human capital accumulation locus and the endogenous growth locus shift up
    ▪ economic growth rises
    ▪ human capital effort (education time) increases as a result of higher \(MPH (1 - u = \sigma(1 - \rho / \phi))\)
    ▪ example: rising year of schooling with reducing work hours

- Generalized goods production function: $F(K, H, \overline{H}) = AK^{1-\beta}(uH)^{\beta}\overline{H}^{\gamma}$
  - $\overline{H}$ captures uncompensated human capital spillover (similar to the argument by Romer 1986) and $\gamma$ measures the degree of spillover
  - $\overline{H}$ is taken as given by individuals and $\overline{H} = H$ in equilibrium
- Main findings:
  - Other than preference parameters, economic growth depends not only on the maximal human capital accumulation rate $\phi$ but on the degree of human capital spillover $\gamma$ (positive growth effect) – higher $\gamma$ lowers $u$ and hence raises the wage rate
  - $H$ grows slower than $K$ (or $Y$ or $C$), explaining stylized fact (S2)
  - Similar to Romer, there is a free rider problem as a result of uncompensated spillover, implying under-investment in education by individuals (inefficiency can be mitigated by education subsidy)
  - Due to uncompensated positive human capital spillover, even the unskilled would migrate to rich countries where the aggregate human capital stock is high and hence the spillover effect is strong, thereby explaining stylized fact (S4).
2. R&D Based Endogenous Growth Model

We begin by developing a basic framework modifying the Nelson-Phelps (1966) technology advancement setup and the Acemoglu-Aghion-Zilibotti (2006) distance-to-frontier setup, which is a simplified version of Wang (2010). We then shift to the horizontal innovation (product variety) model of Romer (1990) and the vertical innovation (product quality) model of Aghion-Howitt (1992).

Remarks:

- Three forms of technological progress: invention, imitation, adoption
  - Datsun/Toyota
  - Hyundai
- Two ways of improving production performance
  - Improve efficiency
  - Change K-L composition

- **Innovation versus implementation:**
  - the leading-edge frontier technology: $\bar{A}$
    - growing on the *quality ladder* at rate $\gamma(n)$, depending on R&D effort $n$
    
    **Note:** the “shape” of the quality ladder depends on
    - a. The size of the leap $\gamma$
    - b. The speed of arrival
    - $\bar{A}$ is taken as given by individuals
  - fraction of sectors on the frontier (innovating sectors): $\eta$
  - fraction of sectors below the frontier (implementing sectors): $(1-\eta)$
  - technology gap: $\bar{A} - A$, with its effect on technical progress depending on implementation/imitation effort $m$
Technology advancement (sector 2):
\[ \dot{A}/A = \eta \gamma(n) + (1 - \eta) \psi(m)(\bar{A} - A)/A \]
Where \( \eta \gamma(n) \) is the growth rate in the frontier sector, and 
\( \psi(m)(\bar{A} - A)/A \) is the growth rate of falling behind firms

- \( \gamma = \gamma_0 n \) captures the frontier technology expansion rate
- \( \psi = \psi_0 m^b, \ 0 < b < 1 \) captures the imitation technology – the effectiveness of imitation depends on the technology gap ratio given by \( (\bar{A} - A)/A = a \)

Lifetime utility: 
\[ U = \int_0^\infty \frac{c^{1-\sigma^{-1}} - 1}{1 - \sigma^{-1}} e^{-\rho t} dt \quad (\text{same as before}) \]
Effective labor: 
\[ L = A(1-n-m) \]
Goods production (sector 1):
\[ F(K, L) = K^{1-\beta} (A(1 - n - m))^\beta \]
Capital accumulation (budget constraint):
\[ \dot{K} = K^{1-\beta} (A(1 - n - m))^\beta - \delta K - c, \ \text{with} \ K(0) = K_0 > 0 \]
• Cross-sector allocation of time implies:
  MPL must be equalized across sectors
  => MPn₁ = MPn₂ and MPm₁ = MPm₂
  => \( \eta \gamma_0 = (1 - \eta) b \psi_0 m^{b-1} a \)

  yielding a positive relationship between the technology gap ratio \( a \) and the implementation/imitation effort \( m \), so \( m = m(a) \), which depends:
  o negatively on frontier technology growth \( \gamma_0 \)
    ▪ need to impose a normality assumption on implementation and goods production
    ▪ as \( \gamma_0 \) increases, \( n \) increases and crowds out both implementation and production time
  o positively on implementation efficiency \( \psi_0 \)
  o negatively on the fraction of sectors on the frontier \( \eta \)
• From the technology advancement equation, we obtain:

\[ \theta = \left[ \frac{\psi_0}{\gamma_0} \left( \frac{1-\eta}{\eta} \right)^b \right]^{1/(1-b)} \]  
(TA locus)

- positive relationship between the technology gap ratio and growth
- advancing technology by imitation is easier when an economy is far below the frontier
- for a given growth rate \( \theta \), higher frontier technology growth \( \gamma_0 \), lower implementation efficiency \( \psi_0 \) or fewer frontier sectors \( \eta \) will enlarge the technology gap ratio \( a \) (i.e., TA locus shifts rightward)

• Similar to other long-run constant-returns models, consumption, output and factor inputs plus the technology (\( c, K, A, \bar{A} \) and \( Y \)) must all grow at the common rate \( \theta = \gamma = \gamma_0 \eta \)

• Keynes-Ramsey:

\[ \theta = \sigma[\eta\gamma_0 - \rho - (1+ba)(1-\eta)\psi_0 m(a)^b] \]  
(KR locus)

- negative relationship between the technology gap ratio and growth
- imitation lowers growth by reducing firms’ incentive to innovate
- for a given technology gap \( a \), higher \( \gamma_0 \), lower \( \psi_0 \) or more frontier sectors \( \eta \) will raise growth \( \theta \) (i.e., KR locus shifts upward)
• Main findings:
  o Both innovation and imitation are valuable: the larger the technology gap ratio \( a \), the more valuable implementation effort \( m \) is
  o Higher frontier growth \( \gamma_0 \)
    ▪ widens the technology gap ratio
    ▪ promotes economic growth \( \theta \)
    ▪ leads to larger firm dispersion
  o Lower implementation efficiency \( \psi_0 \)
    ▪ raises technology gap \( a \)
    ▪ enhances growth \( \theta \)
  o A larger fraction of frontier sectors \( \eta \)
    ▪ increases technology gap \( a \)
    ▪ increase growth \( \theta \)
b. R&D and Horizontal Innovation: Romer (1990)

- Labor allocation: $L_1$ for production and $L_2$ for R&D, with $L_1 + L_2 = L$
- Final good production (numeraire):
  - perfectly competitive
  - produced with labor and a basket of $M$ intermediate goods $x_i$
    - the larger $M$ is, the more sophisticated the production line is
    - the sophistication of the production line can grow, depending on R&D labor: $\dot{M}/M = \lambda L_2$
  - production function: $Y = L_1^{1-\alpha} \int_0^M x_i^\alpha di$
  - labor demand: $MPL_1 = (1 - \alpha)L_1^{-\alpha}Mx^\alpha = w$ (ex post symmetry $x_i=x$)
- Intermediate goods production:
  - monopolistically competitive
  - monopoly rent measured by the markup: $\eta = (1 - \alpha)/\alpha$
  - maximized profit (earned forever with new entry): $\Pi = \eta x$
- R&D decision:
  - profit: $\Pi_R = (\Pi/r_D)\dot{M} - wL_2 = (\Pi/r_D)\lambda L_2 M - wL_2$
  - labor demand: $MPL_2 = (\Pi/r_D)\lambda M = w$
The two labor demand conditions together with maximized intermediate firm profit yield the R&D return per incremental variety:
\[ r_M = \alpha \lambda L_1 \] (RD)

Intermediate varieties growth
(VG): \[ \theta = \frac{\dot{M}}{M} = \lambda L_2 = \lambda (L - L_1) \]
- downward sloping in \( L_1 \)
- higher \( \lambda \) improves R&D efficiency and thus raises intermediate varieties growth

Keynes-Ramsey (KR):
\[ \theta = \sigma(r_M - \rho) = \sigma(\alpha \lambda L_1 - \rho) \]
- upward sloping in \( L_1 \)
- higher \( \lambda \) improves R&D efficiency, raises intermediate firm’s profitability/rent and enhances output growth
- higher markup (lower \( \alpha \)):
  - raises the productivity of R&D, but lowers production labor
  - the former in turn leads to higher growth

Equilibrium: point \( E \) where RD and KR intersect with each other
• Condition for nondegenerate BGP: \( \alpha \lambda L > \rho \)
  o this condition requires that the economy be productive enough such that both sectors are operative
  o this condition is met when:
    - R&D productivity and labor scale are high
    - time discounting and markup are low

• Main findings:
  o higher R&D productivity \( \lambda \) encourages R&D, reallocates labor away from production and raises economic growth (\( E' \))
  o higher \( \alpha \) (lower markup) raises profitability in R&D and increases growth (\( E'' \))
- more intertemporal substitution ($\sigma \uparrow$) or more patient ($\rho \downarrow$) reallocates labor toward R&D and raises growth ($E''$)
- larger employment size ($L \uparrow$) raises production labor, R&D labor as well as growth

- Problem: there is a scale effect – larger countries grow faster, which is unfortunately unrealistic, as pointed out by Jones (1995)

- Vertical innovation: the quality of goods rises with more R&D – the size of the “quality ladder” (QL) is \( \gamma(n) \) and hence technology evolves according to \( \dot{A} = \gamma(n)A \), where \( \gamma(n) = \gamma_0 n^\varepsilon \), with \( 0 < \varepsilon < 1 \)

- Labor allocation:
  - Total non-research labor =N (exogenous)
    - N can be devoted to manufacturing (L) and R&D (n)
  - researcher =R (exogenous) (general researcher, outside of the firm)

- Final good production (numeraire):
  - perfectly competitive
  - produced with only intermediate good
  - production function: \( Y = Ax^\alpha \), \( 0 < \alpha < 1 \)

- Intermediate goods
  - monopoly
  - production function (linear): \( x = L = N - n \)
  - labor demand: \( MPL = \alpha^2 x^{\alpha -1} = w = W/A \) (effective wage)
  - maximized profit: \( \Pi = \eta Axw \), with \( \eta = (1 - \alpha)/\alpha \) measuring the monopoly markup
• Innovator’s decision:
  o *ex ante* perfect competition
  o once innovating successfully and entering the intermediate sector, the innovator becomes an *ex post* monopolist
  o profit: $\Pi_R = \gamma(n)(\Pi/r) - Wn - W_R R$
  o R&D labor demand: $MPL_n = (\Pi/r)\varepsilon\gamma_0 n^{\varepsilon-1} = wA$
    Remark: *ex ante* perfect competition (price taker, here, “value taker” – innovator takes $\Pi$ as given)

• Both labor demand conditions together with maximized profit yield the R&D relationship (RD),
  $r(n) = \eta\varepsilon\gamma_0 n^{\varepsilon-1}(N - n)$, decreasing in $n$ as a result of diminishing returns in R&D

• Endogenous growth rate is driven by vertical innovation, so $\theta = \gamma(n) = \gamma_0 n^\varepsilon$

• Keynes-Ramsey:
  $\theta(n) = \sigma(r - \rho) = \sigma(r(n) - \rho)$
Main findings:
- Economic growth $\theta$ is driven positively by,
  - markup $\eta = (1 - \alpha) / \alpha$
  - labor size $N$ (scale effect)
  - R&D productivity $\gamma_0$
- R&D labor depends positively on markup and labor size
- The effect of $\gamma_0 \uparrow$ on R&D labor:
  - direct incentive effect (+)
  - indirect “R&D labor saving effect” (-)
  - by “normality” arguments, when $\gamma_0 \uparrow$, it is anticipated that $n \uparrow$ (because R&D is more efficient) – this requires that the direct incentive effect dominate the indirect effect

Problem: there is also a counterfactual scale effect with larger countries growing faster.