Choice with Menu-Dependent Rankings (Presentation Slides)

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References that appear on the slides are [1], [2], [3], [4], [5], [6], [7].

References

[1] Markus K. Brunnermeier and Jonathan A. Parker. Optimal expectations. *American Economic Review*, 95(4):1092–1118, September 2005.

[2] M.K. Brunnermeier, C. Gollier, and J.A. Parker. Beliefs in the utility function. Optimal beliefs, asset prices, and the preference for skewed returns. *The American Economic Review*, 97(2):159–165, 2007.

[3] Jon Elster. *Sour Grapes*. Cambridge University Press, 1983.

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[5] Christian Gollier. Optimal illusions and decisions under risk. Working paper, University of Toulouse I, January 2005.

[6] Leonard J. Savage. *The Foundations of Statistics*. John Wiley, New York, 1954.

[7] Ran Spiegler. On two points of view regarding revealed preferences and behavioral economics. Technical report, University College London, September 2006.

Choice with Menu-dependent Rankings

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Standard Theory

- $X \neq \emptyset$ finite set of possible choice objects
- $\mathcal{P}(X) = 2^X \setminus \{\emptyset\}$ all choice situations
- $\mathcal{B} \subset \mathcal{P}(X)$ collection of choice situations
- $c : \mathcal{B} \to \mathcal{P}(X)$ choice function if $c(B) \subset B$ for all $B \in \mathcal{B}$.

Axiom (Weak axiom of Revealed Preference — WARP)

If x and y are both in A and B and if $x \in c(A)$ and $y \in c(B)$, then $x \in c(B)$.

Theorem

The choice function $c : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfies WARP if and only if it maximizes a rational preference relation $\succsim \subset X \times X$.

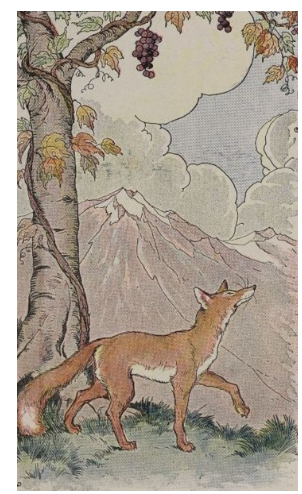
Example (Berger and Smith 1997)

You get a letter asking for a donation...

- $X = \{\$0, \$100, \$500\}$
- $c : \mathcal{P}(X) \to X$
- $c(\{\$0, \$100\}) = \$100$
- $c(X) = \$0$

Fable: The fox and the grapes

"I am sure they are sour."



Two interpretations of Sour Grapes:

- Sore looser
- Jon Elster's (1982) interpretation — preferences underlying a choice may be shaped by the constraints
  - Adaptive preferences
  - Counteradaptive preferences

Second interpretation suggests model with:

- A mapping $A \mapsto \succsim_A$;
- Choice $c(A) = c(A, \succsim_A) = \{x \in A : x \succsim_A y, \forall y \in A\}$

Menus of lotteries

- X Prizes: finite set with elements $x, y, \ldots, z$.
- Δ Lotteries: simplex $\Delta(X)$ of probability measures over X with elements p, q, r.
- $\mathcal{A}$ Menus: closed subsets of $\Delta(X)$ denoted A, B, C.
- $\aleph$ Pairs $(A, p) \in \mathcal{A} \times \Delta$ such that $p \in A$.

The m function

Definition

Let $m : \mathcal{A} \to [0,1]^n$ by

$$m(A) := \left(\max_{p \in A} p(x), \max_{p \in A} p(y), \ldots, \max_{p \in A} p(z) \right)$$

For example,

$$m(\{p\}) = p$$
$$m(\Delta) = (1, 1, \ldots, 1)$$

Also,

$$\|m(A)\|_1 = 1 \iff A = \{p\}.$$

A representation

Definition

A sour grapes representation is a pair (v, a) that consists of a vNM utility index $v \in R^n$ and a constant $a \geq 0$ such that the preference $\succeq$ on $\mathcal{A}$ is represented by the function $W : \mathcal{A} \to \mathbb{R}$ defined by

$$W(A) := \max_{p \in A} \langle v + a \left(\|m(A)\|_1 - 1 \right) m(A), p \rangle$$

Uniqueness

Proposition (uniqueness)

Let (v, a) be a sour grapes representation for the preference $\succeq$ over menus in $\mathcal{A}$. Then (w, b) is another sour grapes representation for the same preference if and only if there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that

$$w = \alpha v + \beta$$
$$b = \alpha a.$$

The Optimal Expectations Model

Brunnermeier & Parker, Gollier (2005, 2005, 2007)

- Time is $\mathbb{T} = \{1, 2, \ldots, T\}$;
- Underlying probability space $(\Omega, \mathcal{A}, \mathbb{P})$;
- State of the world $s : \Omega \times \mathbb{T} \to \{\omega_1, \omega_2, \ldots, \omega_S\}$;
- s_t obtained from s by fixing $t \in \mathbb{T}$;
- Consumption streams $c : \Omega \times \mathbb{T} \to \mathbb{R}_+$;
- Utility index $U : \mathbb{R}_+^T \to \mathbb{R}$ increasing, strictly quasiconcave.

The Optimal Expectations Model

Given a menu of consumption streams A, the agent chooses among the options with the following procedure:

- For each probability measure $\mu \ll \mathbb{P}$ find a consumption stream $c^\mu \in A$ that maximizes $\mathbb{E}_\mu[U(c)]$.
- Choose the measure μ^* that maximizes a well-being index given by

$$\mathcal{W}(\mu) := \mathbb{E}\left[\frac{1}{T}\sum_{t\in\mathbb{T}} \mathbb{E}_\mu\left[U(c^\mu)\middle|s_1, \ldots, s_t\right]\right].$$

Optimal Expectations: consumption example

- $T = 2$
- $\Omega = \{\omega_1, \omega_2, \ldots, \omega_S\}$
- Allocate savings in $t = 1$ and consume in $t = 2$
- One unit endowment
- Two assets: safe with return $R > 0$, risky with return $R + Z$
- $Z_1 < Z_2 < \cdots < Z_S$ and $Z_1 < 0 < Z_S$
- Choose fraction α to invest in safe asset
- Solvency constraint: $R + Z \geq 0 \Rightarrow \alpha \in [-R/Z_S, -R/Z_1]$

Given a belief $\mu \in \Delta(\Omega)$, agent chooses portfolio α^* to maximize $\mathbb{E}_\mu u(R + \alpha Z)$.
Well-being function simplifies:

$$\mathcal{W}(\mu) = \mathbb{E}\left[\frac{1}{T}\sum_{t\in\mathbb{T}} \mathbb{E}_\mu\left[U(c^\mu)|s_1,\ldots,s_t\right]\right].$$
$$= \mathbb{E}\left[\frac{1}{2}\mathbb{E}_\mu[U(c^\mu)|s_1] + \frac{1}{2}\mathbb{E}_\mu[U(c^\mu)|s_1,s_2]\right]$$
$$= \frac{1}{2}\mathbb{E}_\mu[U(c^\mu)] + \frac{1}{2}\mathbb{E}[U(c^\mu)]$$

Motivation
Sour Grapes
Optimal Expectations
Conclusion
Model
Example
Properties

Nice properties:

- Excess optimism and risk taking;
- Preferences for skewness;
- General equilibrium with heterogeneous beliefs and gambling;
- Undersaving and overconfidence.

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Optimal Expectations: properties

Model in terms of choice functions:

- $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space;
- (Z, d) a metric space of prizes;
- A is a set of acts $X : \Omega \to Z$;
- $\mathcal{P}(A) = 2^A \setminus \{\emptyset\}$ all choice situations
- $\mathcal{B} \subset \mathcal{P}(A)$ collection of choice situations
- Choice functions $c : \mathcal{B} \to P(A)$

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Optimal Expectations: properties

Definition

$c : \mathcal{B} \to A$ is an *optimal expectations* choice function if there are $\alpha \in (0,1)$, a utility index $u : Z \to \mathbb{R}$, and a space M of measures on $(\Omega, \mathcal{A})$ including $\mathbb{P}$ such that

- $\forall B \in \mathcal{B}$, $\forall \mu \in M$, $\exists! X^\mu \in B$ that maximizes $\mathbb{E}_\mu u \circ X^\mu$;
- for all $B \in \mathcal{B}$ we have $c(B) = X^{\mu^*}$ where

$$\mu^* \in \arg\max_{\mu \in M} \left\{\alpha \mathbb{E}_\mu u \circ X^\mu + (1-\alpha)\mathbb{E} u \circ X^\mu\right\}.$$

In this case we say that (α, u, M) represents c.

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Optimal Expectations: properties

Proposition (Ran Spiegler, 2008)

Fix $\Omega = \{\omega_1, \dots, \omega_S\}$, the objective measure $\mathbb{P}$ and $\alpha \in (0,1)$. Let X_0 be a safe action, i.e., $u(X_0(\omega)) = 0$ for every $\omega \in \Omega$. Then there exist a material payoff function u and a pair of actions X_1, X_2 such that

$$X_1 \in c(\{X_0, X_1\})$$

and

$$c(\{X_0, X_1, X_2\}) = \{X_0\}.$$

Optimal Expectations: properties

Proposition (Ran Spiegler, 2008 extended)

Fix $\Omega = \{\omega_1, \dots, \omega_S\}$, the objective measure $\mathbb{P}$ and $\alpha \in (0,1)$. Let X_0 be a safe action, i.e., $u(X_0(\omega)) = 0$ for every $\omega \in \Omega$. Then there exist a material payoff function u and a pair of actions X_1, X_2 such that

$$c(\{X_0, X_1\}) = \{X_1\}$$

and

$$c(\{X_0, X_1, X_2\}) = \{X_0\}.$$

Optimal Expectations: properties

Proof.

Construct the following payoff function

Act / state	ω_1	ω_2	$\cdots$	ω_S
X_0	0	0	$\cdots$	0
X_1	1	$-k$	$\cdots$	$-k$
X_2	m	$-n$	$\cdots$	$-n$

where

$$k = \frac{1 + \mathbb{P}(\{\omega_1\})}{1 - \mathbb{P}(\{\omega_1\})} - \varepsilon$$

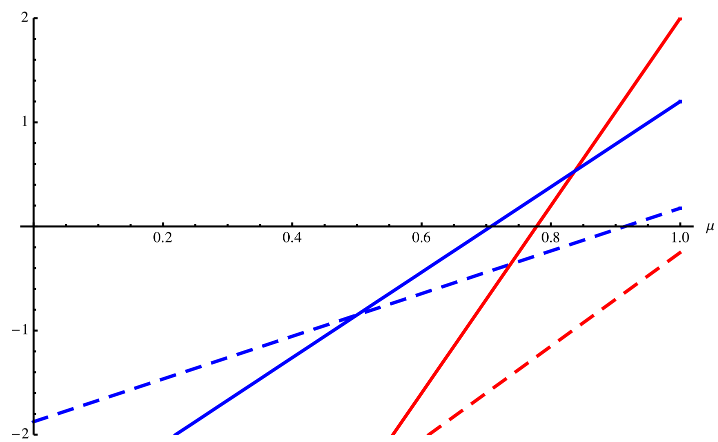
for $\varepsilon > 0$ small, $m > 1$ and $n > (k + \varepsilon)m$. □

Example (Ran Spiegler's extended)

- Three assets and two states of nature
- $\mathbb{P} = (q, 1-q) = (1/2, 1/2)$
- Utility payoffs given by

Asset	payoff in ω_1	payoff in ω_2
X_0	0	0
X_1	1	$-k$
X_2	m	$-n$

When $q = 1/2$, taking $m = 2$, $n = 7$, $k = 2.9$:



Definition

We say that M *preserves independence* if for all $\mu \in M$ (and that includes $\mathbb{P}$) the set of random variables A is an independency.

Proposition 1

An optimal expectations choice function represented by (u, M) where M preserves independence maximizes a rational preference relation $R \subset A \times A$.
Moreover, R admits a Savage subjective utility representation with the state independent utility index given by u and the subjective belief given by a convex combination of the objective measure $\mathbb{P}$ and a special product measure μ^{**}.

The Savage Model

- Set of states of nature Ω
- Prize space Z
- Preference $\succsim$ on the set of acts $X : \Omega \to Z$ satisfying

(A1) weak order
(A2) non-degeneracy
(A3) eventwise monotonicity
(A4) weak comparable probability
(A5) small event continuity
(A6) uniform monotonicity
(A7) sure-thing principle

For details see Fishburn (1970).

The Savage Model

Theorem (Savage)

Axioms A1–A7 imply that there exists a unique, finitely additive, non-atomic probability measure μ and a state-independent utility function $u : Z \to \mathbb{R}$, such that the individual ranks finite-outcome acts $f : \Omega \to Z$ on the basis of

$$\mathbb{E}_\mu u(f) = \int u(f(\omega))d\mu(\omega) = \sum_x u(x)\mu(f^{-1}(x))$$

Proof.

See Fishburn (1970). □

Proof of Proposition 1

1. For each act $X : \Omega \to Z$ in A choose a measure $\mu_X \in M$ that solves
$$\max_{\mu \in M} \mathbb{E}_\mu u \circ X \quad (1)$$
Project μ_X into the σ-algebra generated by X and build the product measure μ^{**} of these projections for all $X \in A$.
2. Take any menu of acts $B \in \mathcal{B}$ and consider the optimal belief $\mu^* \in M$ and the induced choice $X^* \in B$. Note that
$$\mathbb{E}_{\mu^*} u \circ X^* = \mathbb{E}_{\mu^{**}} u \circ X^* \quad (2)$$
since μ^{**} maximizes expression (1) for all X and, in particular, for X^*.

Proof of Proposition 1 (continued)

3. Now we claim that X^* solves
$$\max_{X \in B} \{\mathbb{E}_{\mu^{**}} u \circ X + \mathbb{E} u \circ X\} \quad (3)$$
and we show this in two steps.
4. Take any $X \in B$ that can be induced for some belief $\mu_X \in M$, i.e., where $\mathbb{E}_{\mu_X} u \circ X \geq \mathbb{E}_{\mu_X} u \circ X'$ for all $X' \in B$.
Since M preserves independence, we can take μ_X such that $\mathbb{E}_{\mu_X} u \circ X = \mathbb{E}_{\mu^{**}} u \circ X$. Hence
$$\begin{aligned}\mathbb{E}_{\mu^{**}} u \circ X^* + \mathbb{E} u \circ X^* &= \mathbb{E}_{\mu^*} u \circ X^* + \mathbb{E} u \circ X^* \\ &\geq \mathbb{E}_{\mu_X} u \circ X + \mathbb{E} u \circ X \\ &= \mathbb{E}_{\mu^{**}} u \circ X + \mathbb{E} u \circ X\end{aligned}$$

Proof of Proposition 1 (continued)

5. Pick any $X \in B$ that cannot be induced for any belief $\mu \in M$.
Now consider the measure $\tilde{\mu} \in M$ where the projection of $\tilde{\mu}$ into the sigma algebra generated by any $X' \in B$ is equal to the projection of $\mathbb{P}$, except for the projection into the sigma algebra generated by X, for which it is the same as the projection of μ^{**}.
Since X cannot be induced with $\tilde{\mu}$, there is a $\bar{X} \in B$ such that
$$\mathbb{E}_{\tilde{\mu}} u \circ \bar{X} > \mathbb{E}_{\tilde{\mu}} u \circ X'$$
for all $X' \in B$ and
$$\mathbb{E}_{\tilde{\mu}} u \circ \bar{X} > \mathbb{E}_{\tilde{\mu}} u \circ X$$
Note that $\mathbb{E} u \circ \bar{X} \geq \mathbb{E} u \circ X'$ for all $X' \in B$.

Proof of Proposition 1 (continued)

5. (continued) Therefore
$$\begin{aligned}\mathbb{E}_{\mu^{**}} u \circ X^* + \mathbb{E} u \circ X^* &= \mathbb{E}_{\mu^*} u \circ X^* + \mathbb{E} u \circ X^* \\ &\geq \mathbb{E}_{\tilde{\mu}} u \circ \bar{X} + \mathbb{E} u \circ \bar{X} \\ &> \mathbb{E}_{\tilde{\mu}} u \circ X + \mathbb{E} u \circ \bar{X} \\ &\geq \mathbb{E}_{\tilde{\mu}} u \circ X + \mathbb{E} u \circ X \\ &= \mathbb{E}_{\mu^{**}} u \circ X + \mathbb{E} u \circ X\end{aligned}$$
so X^* in fact solves problem (3).

Proof of Proposition 1 (continued)

6. This means that c maximizes the utility index

$$U(X) = \mathbb{E}_{\mu^{**}} u \circ X + \mathbb{E} u \circ X$$

and of course it will also maximize

$$\frac{1}{2} U(X) = \frac{1}{2} \left[\mathbb{E}_{\mu^{**}} u \circ X + \mathbb{E} u \circ X \right] = \mathbb{E}_{\left(\frac{1}{2}\mu^{**} + \frac{1}{2}\mathbb{P}\right)} u \circ X$$

and this is just an expected subjective utility representation, where the subjective measure is a convex combination of μ^{**} and $\mathbb{P}$. □

Summary

- Motivation for relaxations of WARP;
- Sour Grapes suggests model $A \mapsto \succsim_A$;
- Example — Optimal Expectations;
- Optimal Expectations violates WARP;
- OE + Independence = Savage;

Discussion and next steps

- Relation between SG and OE?
- Optimal Expectations and comonotonicity?
- Optimal Expectations in mixture space?
- Theories of weakening of WARP?
- Complete and transitive preference over *menus*?
- A Theory of Sour Grapes