



Managing the maturity structure of government debt[☆]

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Abstract

It is known that a government can implement the optimal complete-market Ramsey allocations by issuing non-contingent bonds of different maturities. The implied optimal maturity structure is time- and state-invariant—i.e. it is not actively managed. I construct a model where the Ramsey allocations can be implemented with active management of the maturity structure. In a numerical example that reflects the time-series properties of the British government's expenditure during the 18th century, its historic pattern of maturity management is replicated.

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1. Introduction

Angeletos (2002) and Buera and Nicolini (2004) show how the optimal allocations of Lucas and Stokey (1983) can be implemented without state-contingent bonds. Their insight is that the relative price of two risk-free bonds of different maturities depends on the state of the economy. For example, a risk-free one-period discount bond issued at t , whose par value is one unit of consumption good, is worth one unit of consumption good at $t + 1$ no matter what. However, at $t + 1$, a risk-free two-period discount bond issued at t is now

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equivalent to a risk-free one-period discount bond issued at $t + 1$, whose value depends on the realization of the state that affects the interest rate. Thus, by carefully choosing its maturity structure, a government can span the pay-off space of state-contingent claims.

One intrinsic feature of Angeletos (2002) and Buera and Nicolini (2004) is that the government generally needs as many bonds of different maturities as there are possible states. Another is that the optimal maturity structure is time- and state-invariant.

In reality, governments issue a few bonds of different maturities and then actively manage their maturity structure. Here I construct a model whose optimal maturity structure is compatible with this fact. I begin with the general case of Angeletos or Buera and Nicolini where the government expenditure follows an N -state Markov chain. Now assume that, although there are N possible states, starting from a given state, the number of states that can be reached *in one period* is at most N_1 , where $N_1 < N$. This restriction implies that the transition matrix of this N -state Markov chain can have at most N_1 non-zero elements in each row, even though it may well be irreducible—that is, starting from any state, all N states can be reached *eventually* with probability one.¹ For the purpose of implementing the complete-market Ramsey allocations, one option for the government is to issue N bonds of different maturities and form a time- and state-invariant maturity structure. Alternatively, the government may issue only N_1 bonds of different maturities while dynamically re-balancing its portfolio. If there is positive cost associated with creating new securities, the government will choose the latter option.

One well-known example of active maturity management is the British government's debt policy during the 18th century. In periods of low expenditure, the government issued long-term bonds (consols, in particular) while maintaining a sinking fund—a war chest of liquid assets that can be redeemed on short notice. When a war broke out and its expenditure went up, the government issued short-term bonds, which were to be retired and consolidated into long-term bonds once the war was over (Steuart, 1767, Part IV).

In a numerical example, I approximate the British government's expenditure series during the 18th century with a Markov chain that has $N = 3$ and $N_1 = 2$. The three states represent peacetime, early stages of a war, and late stages of a war. The transition matrix is constructed to capture an asymmetric pattern in the data (Fig. 1): the expenditure rises gradually as a war intensifies, but falls abruptly once it ends. It is further assumed that there are only two non-contingent bonds of different maturities. To implement the complete-market Ramsey outcome, the government has to form different maturity structures for each state realization—in a way that mimics the British practice.

2. The economy

The economy is populated by a continuum of identical households of measure one. Households order their consumption–leisure allocation streams by $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t, x_t)$, where c and x stand for consumption and leisure. u is strictly concave, increasing and twice continuously differentiable in both arguments. Furthermore, u is assumed to be additively separable in the two arguments.

Households are endowed with ξ units of perfectly divisible time each period, which can be either consumed as leisure or devoted to production. Labor is the only input of production, and the production technology exhibits constant returns to scale: $y_t = \xi - x_t$.

¹The difference between N and N_1 will be particularly large for highly persistent processes.

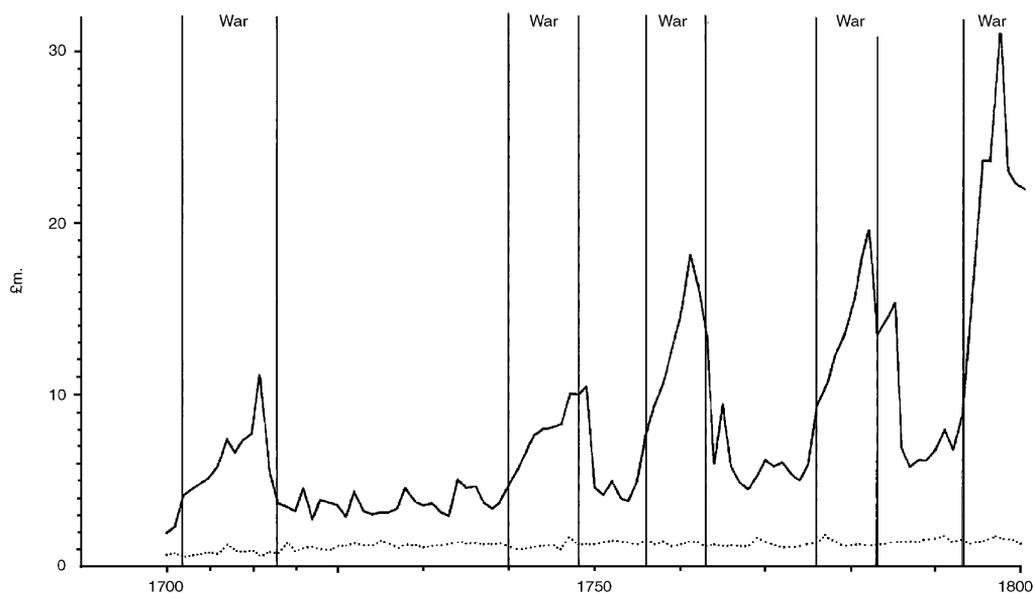


Fig. 1. The British government's annual expenditure in the eighteenth century (1701 prices). Reproduced from Jackson (1990, p. 222) whose data source is Deane and Mitchell (1962). The solid line is the total government expenditure, and the dotted line is civil expenditure. Note that the periods with high government spending overlap with wartime.

There is a government in the economy that maximizes the welfare of the households. Its welfare criterion is the same as the representative household's objective function. The government is a Ramsey planner. At $t = 0$, it announces and commits to a system of prices and taxes over all possible history realizations.

There is a flat-rate tax on labor income, and it is the only source of tax revenue. The model rules out any lump-sum transfer between the households and the government.

In addition, the government creates a bond market to implement complete-market allocations and decides on which bonds will be traded in terms of maturity. Both the government and the households are allowed to issue and purchase bonds in any quantity as long as their intertemporal budget constraints are met.

Government expenditure, g_t , follows an exogenous stochastic process, and it is the only source of uncertainty in this economy. More specifically, g_t -process is modeled with an N -state Markov chain, whose realizations are publicly observable. Let s_t denote the state at t : $s_t = g_t$, $s_t \in \mathcal{S}$, where \mathcal{S} is the state space. The history up to t is denoted by s^t , $s^t \in \mathcal{S}^{t+1}$. It is assumed that g_t does not enter the households' utility function.

3. Ramsey allocations with complete markets

The standard algorithm for solving a Ramsey problem is to substitute out prices with allocations using households' optimality conditions. Once the problem is re-written in such primal form, the optimal complete-market allocations can be determined without reference to the specifics of the asset market. Then, solving for an optimal portfolio is simply a matter of finding one that supports the optimal allocations in the equilibrium.

The complete-market Ramsey problem in primal form is to maximize the welfare criterion subject to the implementability constraint:

$$\begin{aligned} & \max_{\{c_t(s^t), x_t(s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) u(c_t(s^t), x_t(s^t)) \\ \text{s.t. } & \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) \left[\frac{u_{c,t}(s^t)}{u_{c,0}} c_t(s^t) - \frac{u_{x,t}(s^t)}{u_{c,0}} (\xi - x_t(s^t)) \right] = b_0, \end{aligned} \quad (1)$$

where u_c and u_x denote the derivatives of u with respect to c and x , and $\pi(s^t)$ is the probability of s^t . The value of the initial debt that the government begins with is b_0 . The implementability constraint (1) is obtained by converting the intertemporal budget constraint of the government at $t = 0$ into primal form, and then by substituting out g_t using the resource constraint: $c_t + g_t = \xi - x_t$. The household budget constraint becomes redundant once the government budget constraint and the resource constraint are satisfied. A more detailed derivation can be found in Lucas and Stokey (1983, p. 61). With complete markets, the above infinite-dimensional problem can be further simplified, because the optimal allocations are not history dependent.

4. Optimal maturity structure of non-contingent bonds

As in Angeletos (2002) and Buera and Nicolini (2004), the optimal maturity structure can be reverse-engineered from the optimal allocations. In particular, to support the optimal allocations in the equilibrium, the optimal maturity structure must satisfy the sequential implementability constraints at each history node.

For example, assume that there are M risk-free discount bonds, whose maturities run sequentially from 1 to M . Define the value at t of the government's outstanding obligations, denominated in time- t consumption unit, by $B(s_t|s^{t-1}) := \sum_{j=0}^{M-1} p_t^{t+j} b_t^j$, where p_t^{t+j} is the price at t of a risk-free discount bond that delivers one unit of consumption good at $t+j$. b_t^j is the quantity of bonds whose remaining maturity at t is j . Note that b_t^j is determined at $t-1$. Using households' optimality conditions to substitute out the bond prices,

$$B(s_t|s^{t-1}) = \sum_{j=0}^{M-1} \beta^j \frac{\mathbb{E}_t u_{c,t+j}}{u_{c,t}} b_t^j.$$

The present discounted value of the government's surplus stream can be defined as

$$z(s_t|s^{t-1}) := \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j \left(\frac{u_{c,t+j}}{u_{c,t}} c_{t+j} - \frac{u_{x,t+j}}{u_{c,t}} (\xi - x_{t+j}) \right) \right].$$

Given the Markov property of the g_t -process, $z(s_t|s^{t-1})$ and $B(s_t|s^{t-1})$ depend only on s_t . The sequential implementability condition for each realization of s_t is then $z(s_t) = B(s_t)$. When $M \geq N$, the complete-market Ramsey allocations can be supported. However, only when $M = N$, the optimal maturity structure is uniquely determined. See Angeletos (2002) or Buera and Nicolini (2004) for more detail.

Typically, such an optimal maturity structure is time- and state-invariant—that is, there is no active maturity management. Empirically, however, maturity structures differ over

time and across states. Eighteenth-century Britain, as mentioned in the introduction, provides a good example.

Here I construct a complete-market Ramsey problem that entails state-dependent maturity structures. This example reflects the actual expenditure series of the British government in the 18th century as shown in Fig. 1. In particular, I approximate the government expenditure with a Markov chain with $N = 3$ and $N_1 = 2$ at yearly frequency:

$$\begin{bmatrix} g_L \\ g_M \\ g_H \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.2 \\ 1.9 \end{bmatrix}, \quad \Pi = \begin{bmatrix} \pi_L & 1 - \pi_L & 0 \\ 0 & \pi_M & 1 - \pi_M \\ 1 - \pi_H & 0 & \pi_H \end{bmatrix},$$

where $\pi_L = 0.9$, $\pi_M = 0.6$ and $\pi_H = 0.6$. The transition matrix (Π) captures the asymmetry in the speed with which the government expenditure rises and falls. The expenditure, as depicted in Fig. 1, rises gradually ($g_L \rightarrow g_M \rightarrow g_H$), but then falls abruptly to pre-war levels at the conclusion of a war ($g_H \rightarrow g_L$). The non-zero elements of Π are chosen to match the average duration of wars (about 7 years) and that of inter-war periods (about 13 years). I let $u(c_t, x_t) = \log c_t + \psi \log x_t$, $\psi = 1.5$, $\zeta = 10$ and $\beta = 0.95$. The level of g 's are chosen to match the ratio of government spending to total output reported by Deane and Mitchell (1962). For example, the government spending amounts to about 13% of total output during peacetime ($g_t = g_L$). It is also assumed that $g_0 = g_L$ and $b_0 = 0$.

In this example, the government issues one-period bonds and consols (perpetuities that pay $1 - \beta$ units of consumption good for ever). Because there are only two non-zero elements in each row of Π , the complete-market outcome can be implemented with two securities if the government dynamically re-balances its portfolio.² For example, when $g_t = g_L$, g_{t+1} can only be either g_L or g_M . Therefore, the government will choose the quantity of one-period bonds (b_1) and consols (b_c) so that $B(g_{t+1} = g_L) = z(g_{t+1} = g_L)$ and $B(g_{t+1} = g_M) = z(g_{t+1} = g_M)$, which is simply a matter of solving two linear equations in two unknowns.³ The above parameter values yield $B(g_L) = 0$, $B(g_M) = -0.87$ and $B(g_H) = -0.72$. Obviously, $B(g_L) = 0$ follows from the assumption of $g_0 = g_L$ and $b_0 = 0$.

The implied optimal maturity structure is state-contingent (Table 1). In peacetime, the government accumulates one-period bonds and issues consols ($b_1 = -14.36, b_c = 13.80$). This is consistent with the state-invariant optimal maturity structure of Angeletos or Buera and Nicolini. However, when a war breaks out ($g_t = g_M$), the optimal maturity structure of the government consists of issuing one-period bonds and holding consols ($b_1 = 0.76, b_c = -1.67$). Once the war reaches its climax ($g_t = g_H$), the government reverses its asset position to hold one-period bonds and issue consols ($b_1 = -4.83, b_c = 4.64$).

The reason why the typical optimal maturity structure has $b_1 < 0$ and $b_c > 0$ is as follows. To smooth taxes across states, the government will hold a portfolio that increases in value exactly when its expenditure is higher. When g_t is high, households' marginal

²Alternatively, if the government issues one additional security whose pay-off is linearly independent of one-period bonds and consols, it can form an optimal maturity structure that is time- and state-invariant.

³See Buera and Nicolini (2004, p. 538) for details on how this system of linear equations can be formulated. Note that the cum-dividend price of the consol at the beginning of $t + 1$ —but after the realization of s_{t+1} —is

$$(1 - \beta) \frac{\sum_{j=1}^{\infty} \beta^{j-1} \mathbb{E}[u_{c,t+j}(s_{t+j}) | s_{t+1}]}{u_{c,t+1}(s_{t+1})}.$$

Table 1
Optimal maturity structure of government debt in each state

b_{1L}	b_{cL}	B_L	y_L	b_{1M}	b_{cM}	B_M	y_M	b_{1H}	b_{cH}	B_H	y_H
-14.36	13.80	0.0	3.80	0.76	-1.67	-0.87	4.23	-4.83	4.64	-0.72	4.66

b_1 is the quantity of one-period bonds issued by the government, and b_c is the quantity of consols. A negative number indicates that the government is holding bonds issued by households. The value of the government's portfolio in each state is $-B$. The output (y) in each state is reported alongside. The subscripts L , M and H denote the realization of government expenditure.

Table 2
Optimal maturity structure for different persistence parameter values

π_L	π_M	π_H	b_{1L}	b_{cL}	y_L	b_{1M}	b_{cM}	y_M	b_{1H}	b_{cH}	y_H
0.9	0.6	0.5	-11.46	11.05	3.82	1.11	-1.94	4.24	-4.02	3.87	4.68
0.9	0.6	0.7	-16.00	15.25	3.77	0.13	-1.10	4.21	-5.52	5.26	4.62
0.8	0.5	0.7	-10.78	10.04	3.66	-0.62	-0.10	4.09	-4.53	4.22	4.52

Recall that the baseline case (Table 1) has $\pi_L = 0.9$, $\pi_M = 0.6$ and $\pi_H = 0.6$.

utility of consumption goes up and so does the real interest rate. Hence the relative price of long-term bonds to one-period bonds falls—a one-period bond at maturity is always worth one unit of consumption good. With $b_c > 0$, the government's liability falls in value (e.g. from $B(g_L) = 0$ to $B(g_M) = -0.87$) and the value of its portfolio rises, allowing the tax rate to be lower than what it would have been otherwise. This maturity structure is consistent with the fact that the British government issued long-term bonds and maintained a sinking fund during peacetime.

Then why does the optimal maturity structure at the beginning of a war ($g_t = g_M$) involve $b_1 > 0$ and $b_c < 0$, when the next period's government expenditure may well be even higher ($g_H > g_M$)? The answer lies with $z(s_t)$, the present value of the government's future surplus stream. Recall that the optimal maturity structure with complete markets is chosen such that $z(s_t)$ is equal to $B(s_t)$, the time- t value of the government's outstanding liability, for each s_t . When $g_t = g_M$, even though g_M is less than g_H , the government is in the worst situation in terms of $z(s_t)$, because the next period expenditure can only be either g_M or g_H . On the other hand, once g_H occurs and the war reaches its climax, there is a good chance that the war will conclude soon— g_L will be realized in the next period—as long as the g_H -state is not too persistent. In summary, $z(g_M) < z(g_H) < 0$ because the present value of the future government deficit ($-z$) is expected to be higher with g_M than with g_H . Thus, the optimal portfolio of the government must decrease commensurately in value when g_H occurs following g_M , which is attained with $b_1(g_M) > 0$ and $b_c(g_M) < 0$.

Table 2 reports the optimal state-contingent maturity structure for alternative parameterizations of Π . The qualitative results remain the same, except for the last row, where the g_H -state is much more persistent compared to the g_M -state.

One shortcoming of this class of models, as pointed out by Buera and Nicolini (2004), is that the implied portfolio positions can be very big, which applies to this example as well. When $g_t = g_L$, the optimal maturity structure consists of portfolio positions that are

almost four times the output. However, the positions have opposite signs, and the value of the portfolio is no more than one fifth of the output in the numerical example.

5. Concluding remarks

I adapt the model of Angeletos (2002) and Buera and Nicolini (2004) to make it compatible with the empirical fact that governments issue a few bonds of different maturities and actively manage their maturity structure. A calibrated version of the model replicates the debt management pattern of the 18th-century British government.

One conclusion is that dynamic re-balancing allows governments to implement complete-market Ramsey outcomes with fewer securities than previously thought. This further points to the possibility that, even when the government expenditure is a continuous random variable, the government may still be able to implement complete-market outcomes by continuously re-balancing its portfolio composed of a few bonds.⁴ This will be an application of the results in Duffie and Huang (1985).

The analysis here relies on the assumption of complete markets. It will be interesting to explore how the maturity structure should be managed with incomplete markets.⁵ This is a topic for future research.

References

- Angeletos, G.-M., 2002. Fiscal policy with non-contingent debt and the optimal maturity structure. *Quarterly Journal of Economics* 117, 1105–1131.
- Buera, F., Nicolini, J.P., 2004. Optimal maturity of government debt without state contingent bonds. *Journal of Monetary Economics* 51, 531–554.
- Deane, P., Mitchell, B.R., 1962. *Abstract of British Historical Statistics*. Cambridge University Press, Cambridge.
- Duffie, D., Huang, C.-F., 1985. Implementing Arrow-Debreu equilibria by continuous trading of few long-lived securities. *Econometrica* 53, 1337–1356.
- Jackson, R.V., 1990. Government expenditure and British economic growth in the eighteenth century: some problems of measurement. *Economic History Review* 43, 217–235.
- Lucas Jr., R.E., Stokey, N.L., 1983. Optimal fiscal and monetary policy in an economy without capital. *Journal of Monetary Economics* 12, 55–93.
- Sargent, T.J., Velde, F.R., 1995. Macroeconomic features of the French Revolution. *Journal of Political Economy* 103, 474–518.
- Steuart, J., 1767. *An Inquiry into the Principles of Political Economy*, vol. 4. Millar and Cadell, London.

⁴Note that the theoretical results of Angeletos (2002) and Buera and Nicolini (2004) are obtained only for the cases where randomness has finitely many possible outcomes.

⁵In their analysis of the British public finance during the 18th century, Sargent and Velde (1995) observe that tax rates and debt services are much more persistent than government expenditure. They conclude that incomplete-market models may be a better description of these historic facts than complete-market ones.